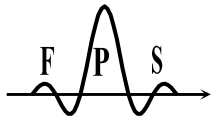


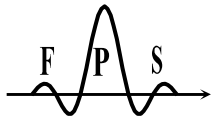
Summary (1/2)

- *Characterization and representation of discrete signals*
 - *Types of signals*
 - *Discrete-time signals*
 - *representation of a discrete-time signal*
 - *basic discrete-time signals*
- *Characterization and representation of discrete systems*
 - *Properties of discrete-time systems*
 - *Linear time-invariant systems (LTI)*
 - *response to a discrete-time input*
 - *Discrete-time convolution*
 - *properties of LTI systems*
 - *FIR and IIR systems*
 - *(linear) difference equations with constant coefficients*



Summary (2/2)

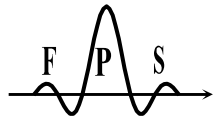
- *Basic signal properties*
 - *Continuous-time versus discrete-time*
 - *Periodic versus aperiodic*
 - *Deterministic versus random*
 - *Energy versus power*
 - *Sinusoidal sequences with a prescribed SNR*
- *the auto-correlation and the cross-correlation*
 - *concept and meaning*
 - *definition of the auto-correlation*
 - *definition of the cross-correlation*
 - *auto-correlation and cross-correlation examples*
 - *auto-correlation and cross-correlation properties*



Characterization and representation of discrete signals

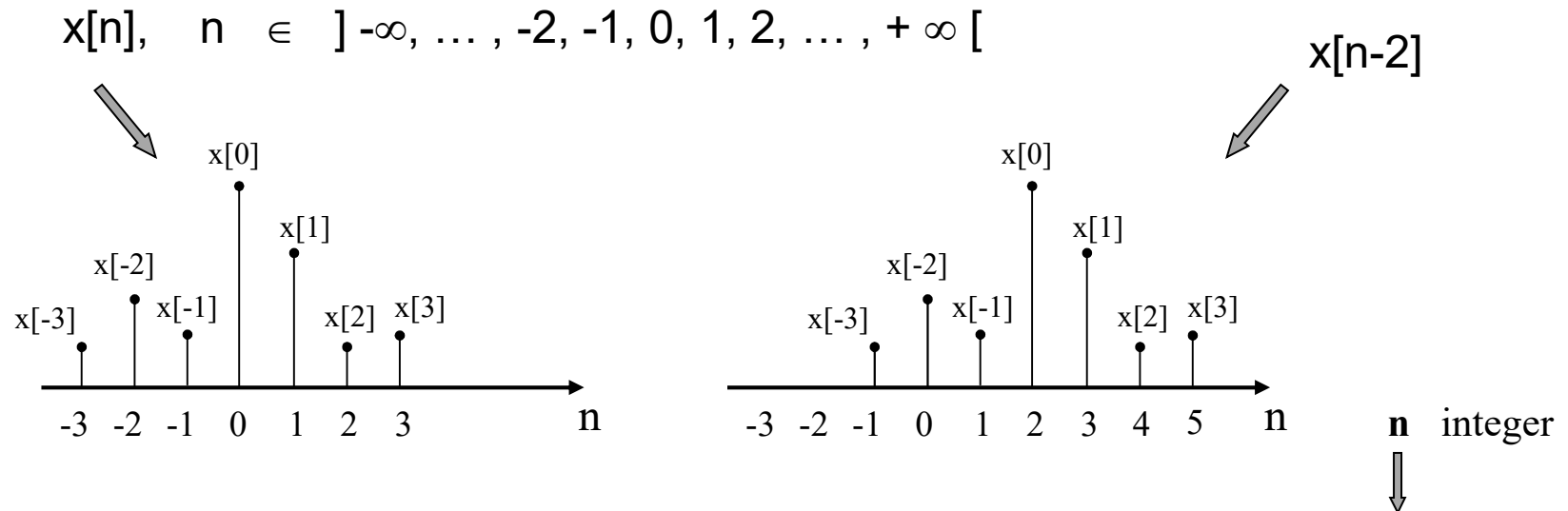
- types of signals and systems
 - **continuous-time signal** (or analog) = continuous function of independent continuous-time variables
 - **continuous-time systems**: those whose inputs and outputs are continuous
 - **discrete-time signal** = continuous function of independent discrete-time variables
 - **digital signal**: discrete function of independent discrete-time variables
 - **digital systems**: those whose inputs and outputs are discrete

Despite the fact that digital signal processing presumes digital (*i.e.*, quantized) signals, we will focus mainly on discrete signals and systems and will address quantization separately, when that is required.



Characterization and representation of discrete signals

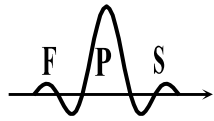
- representation of a discrete-time signal



$x[n]$ represents symbolically the ENTIRE discrete signal for $-\infty < n < +\infty$

taking a specific value of n , for example $n=2$, then $x[2]$ represents the magnitude of the sample of $x[n]$ at position 2 in the sequence of numbers

$x[n-n_0]$ represents $x[n]$ delayed by n_0 samples (admitting n_0 is positive)

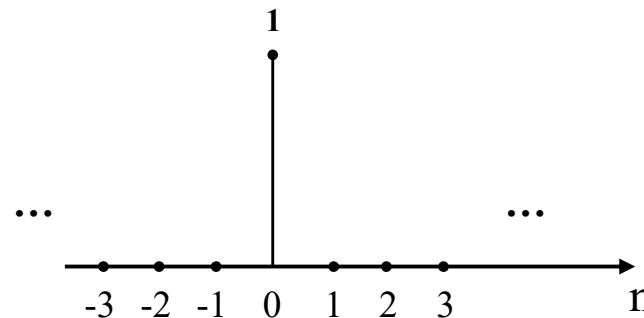


Characterization and representation of discrete signals

- basic discrete-time signals

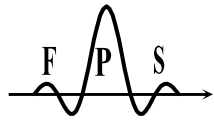
unit impulse (and not “Dirac” impulse! – what is the difference ?)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



- this signal is very important for various reasons, including the possibility to express any discrete-time sequence as a sum of scaled, delayed impulses :

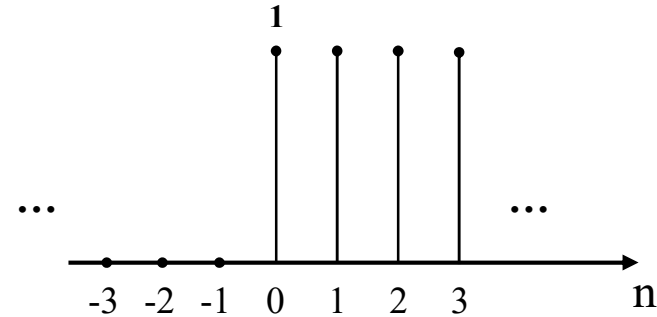
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k]$$



Characterization and representation of discrete signals

unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



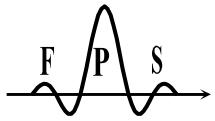
– it is possible to write:

$$u[n] = \sum_{k=-\infty}^n \delta[k] = \sum_{k=0}^{+\infty} \delta[n-k]$$

accumulated sum
of impulses
till position n

sum of
delayed impulses

NOTE: it is also possible to write $\delta[n] = u[n] - u[n-1]$

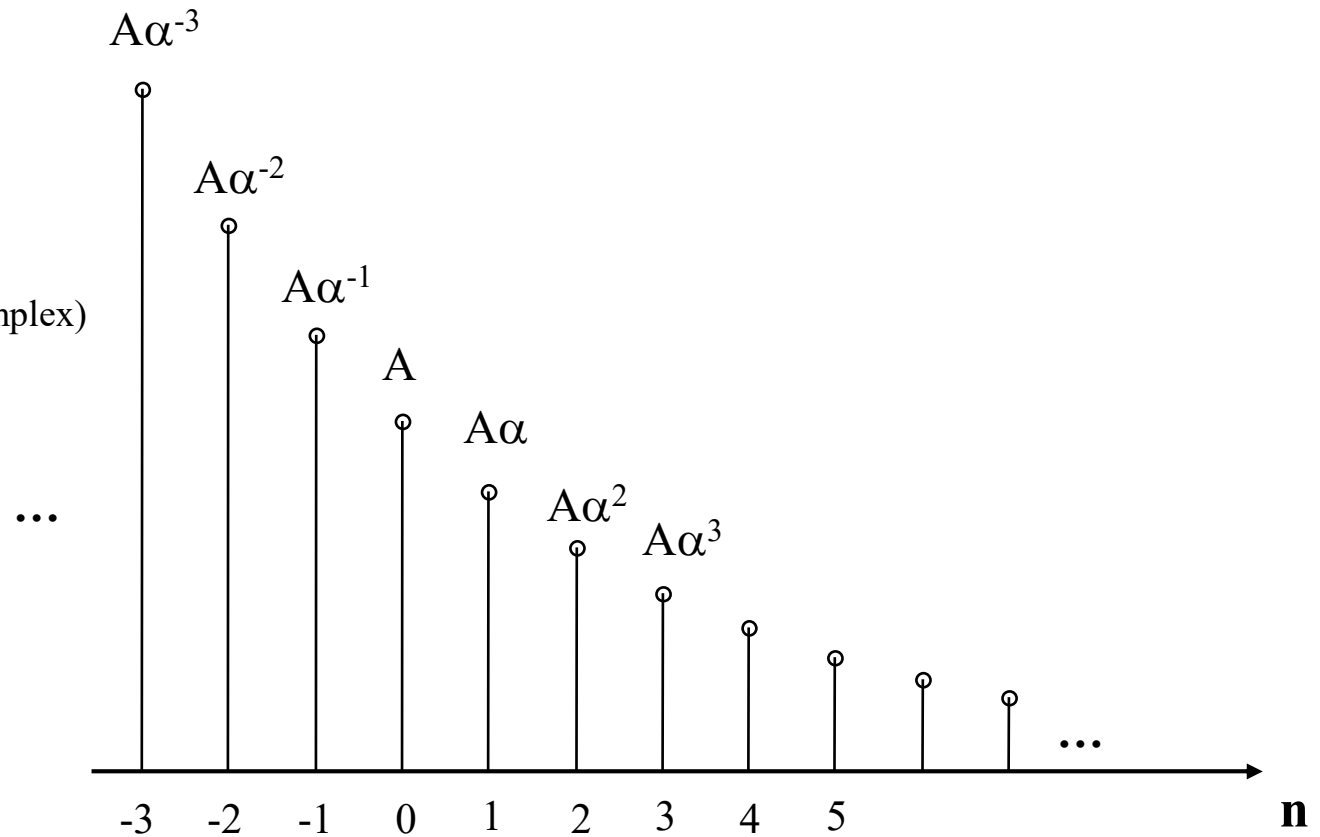


Characterization and representation of discrete signals

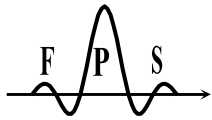
exponential sequences

$$x[n] = A\alpha^n$$

(A and/or α may be complex)



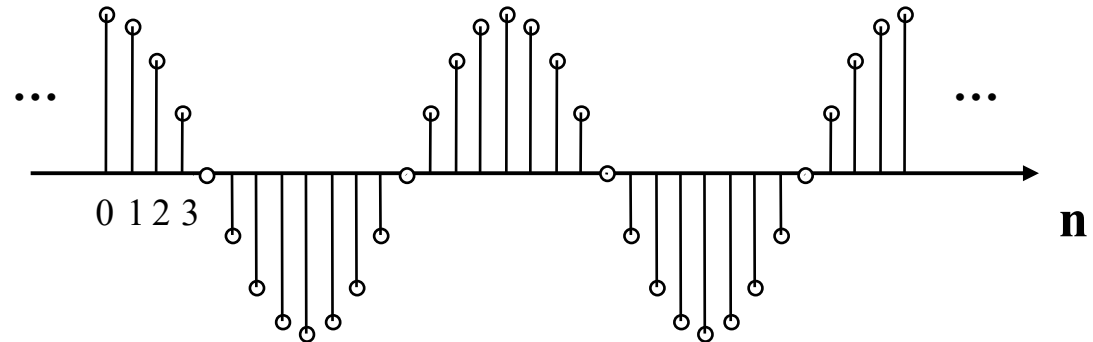
important particular case: $A=1$ and $\alpha=e^{j\omega}$ (unit complex exponential)



Characterization and representation of discrete signals

sinusoidal sequences

$$x[n] = A \cos(n\omega_0 + \phi)$$



ω_0 - frequency of the sinusoidal sequence [radians]

ϕ - phase of the sinusoidal sequence [radians]

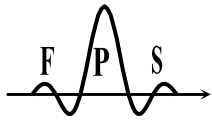
NOTE: since $A \cdot \cos[n(\omega_0 + k2\pi) + \phi] = A \cdot \cos[n\omega_0 + \phi]$ with k integer, frequency ω_0 is only defined, for example, in the range $]-\pi, +\pi]$ or $[0, 2\pi[$

combination of sequences

It is common to combine basic discrete-time sequences to express a large variety of signals, e.g. :

$$x[n] = A\alpha^n u[n]$$

$$x[n] = A(u[n] - u[n - n_0])\alpha^n$$

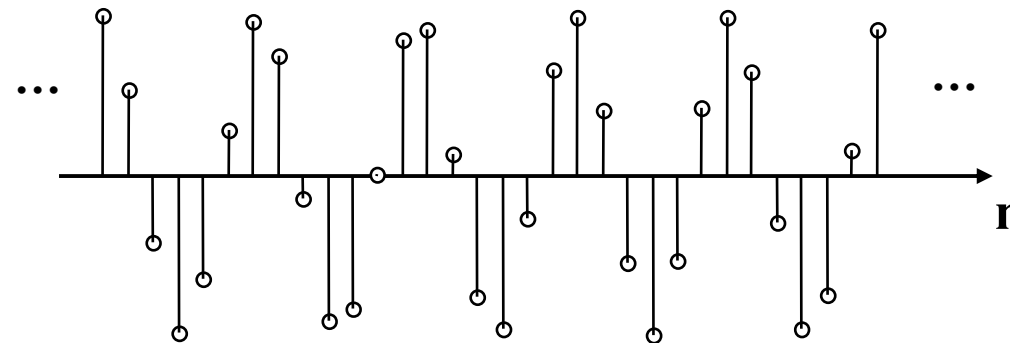


Characterization and representation of discrete signals

periodic sequences

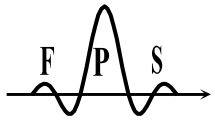
those verifying $x[n]=x[n+N]$, for a given integer N and for any value of n : $-\infty < n < +\infty$

NOTE: since the n index may only take integer values, the counter-intuitive case may occur of a function, such as $\cos(n\omega_0+\phi)$, not showing periodicity in n , for a given ω_0 , as for example $\cos(n)$:



QUESTION: Under what condition is that ω_0 insures periodicity in n ?

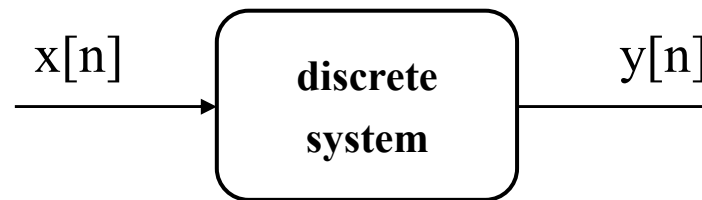
A : $\cos(n\omega_0+\phi) = \cos(n\omega_0+N\omega_0+\phi)$, $\Rightarrow N\omega_0 = k2\pi \Leftrightarrow \omega_0=2\pi k/N$



Characterization and representation of discrete systems

- Discrete-time systems

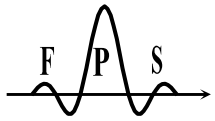
- they transform an input discrete-time sequence into an output discrete-time sequence



- Example 1 (delay) : $y[n]=x[n-n_d]$, n_d positive integer = system delay

- Example 2 (moving average) :
$$y[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} x[n-k]$$

the output at position n is the average of the (N_1+N_2+1) input samples between position $n-N_2$ and position $n+N_1$



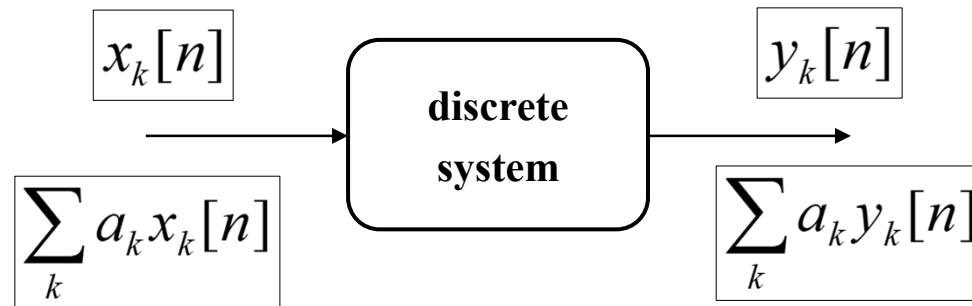
Characterization and representation of discrete systems

- Properties of discrete-time systems

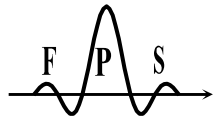
memory - a memoryless system depends only on the input sample at position n to generate an output sample at the same position

- example: $y[n] = (x[n])^2$

linearity – a linear system complies with the superposition principle

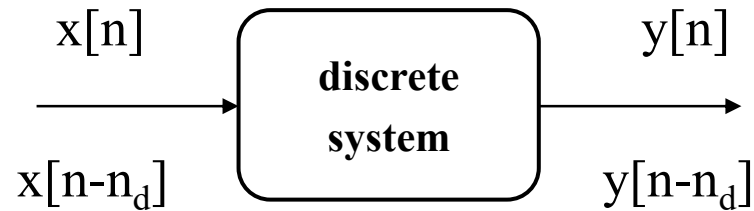


- example: the “accumulator” function $y[n] = \sum_{k=-\infty}^n x[k]$ is linear
- example: function $y[n] = \log_{10}|x[n]|$ is non-linear



Characterization and representation of discrete systems

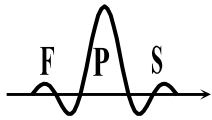
time invariance – a system is time-invariant if a delay of the input sequence gives rise to the same delay of the output sequence



- example: the “accumulator” $y[n] = \sum_{k=-\infty}^n x[k]$ is time-invariant because if the input samples are delayed by n_0 , so are the output samples: $y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$
- example: the decimator system $y[n] = x[nM]$ is not time-invariant

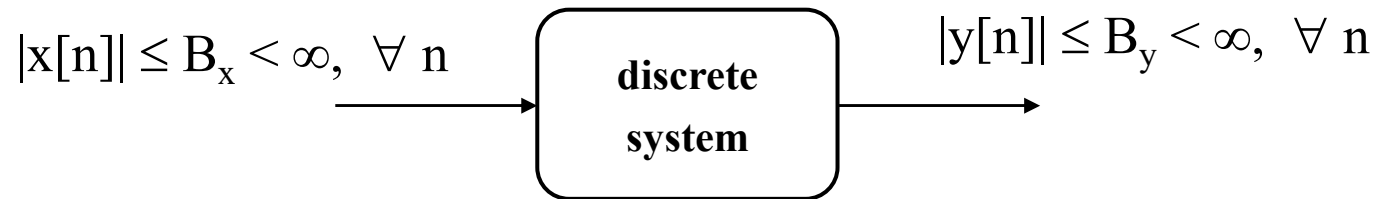
causality – a system is causal (*i.e., it is non-anticipative*) if for any n_0 , the output of the system at position $n=n_0$ depends uniquely on the input samples for $n \leq n_0$

- example: $y[n] = x[n] - x[n-1]$ is causal
- example: $y[n] = x[n+1] - x[n]$ is not causal



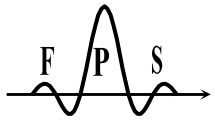
Characterization and representation of discrete systems

stability – a system is stable if any bounded input sequence gives rise to an output sequence that is also bounded



- example: the system $y[n] = (x[n])^2$ is stable
- example: the system $y[n] = \log_{10} |x[n]|$ is not stable (e.g., for $x[n]=0$)
- example: the system $y[n] = \sum_{k=-\infty}^n x[k]$ is not stable (e.g., for $x[n]=u[n]$)

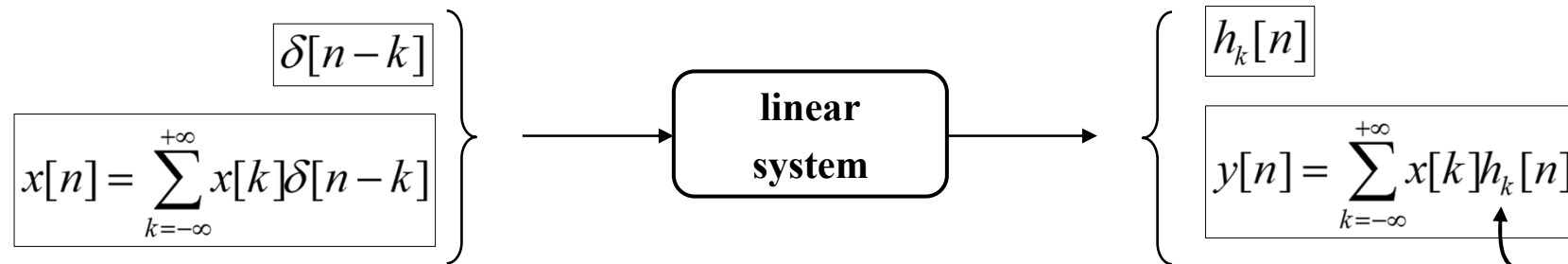
NOTE: a sufficient proof of instability is to find/show a case not complying with the stability condition



LTI systems: response to a discrete input

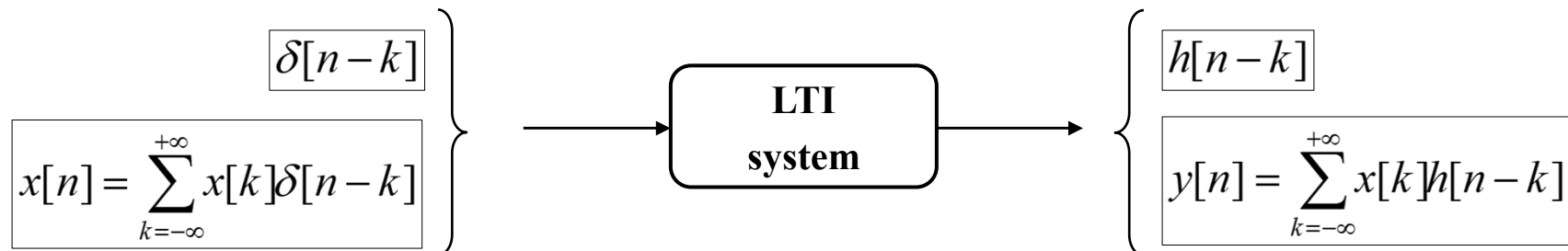
- response to a discrete-time input

- if the impulse response is not time-invariant, we have:

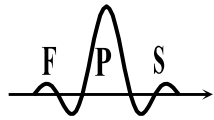


this result is of limited usefulness since the response of the system to a linear combination of input impulses is the same combination of the individual responses to the input impulses which may depend on the position of the impulses (time variance)

- now, if we consider time-invariance:



the output of an LTI system is expressed as a function of a single impulse response !



Linear time-invariant systems (LTI)

- we may thus say that an LTI system is completely characterized by its impulse response $\Leftrightarrow$ given $h[n]$, it is possible to know the response of the LTI system to *any* input:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

this equation consists in the discrete-time convolution or, in other words, the convolution sum which is reminiscent of the familiar convolution integral for continuous-time signals:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

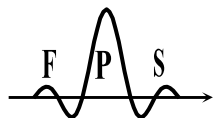
- discrete-time convolution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

independent variable

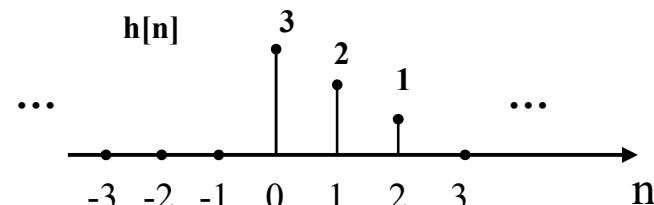
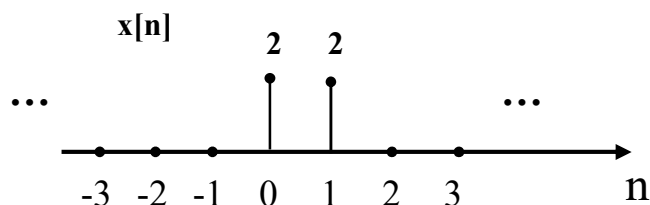
independent variable

parameter (shift in k) !

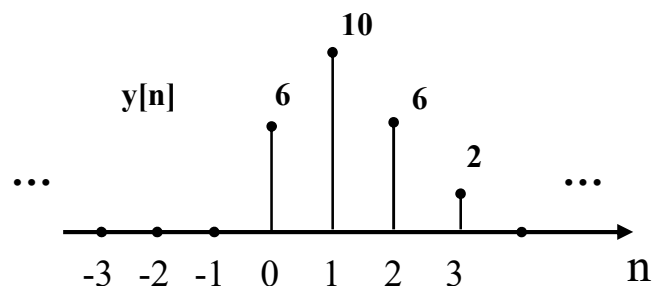
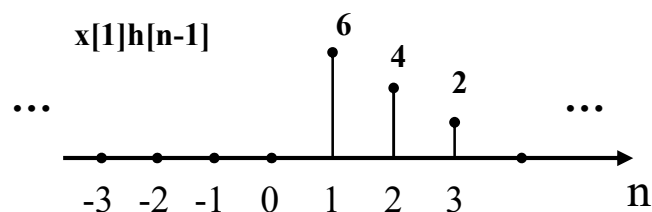
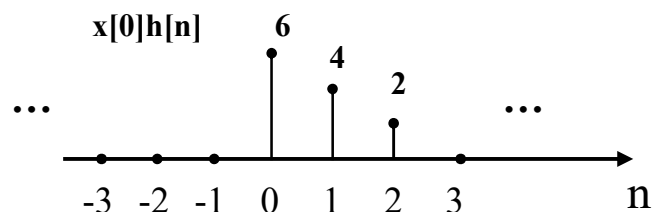


LTI systems: the discrete convolution

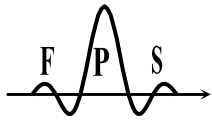
example of the discrete convolution between two sequences:



- Method 1: solving by realizing k , we obtain a weighted sum of impulse responses: $y[n]=x[0]h[n]+x[1]h[n-1]$

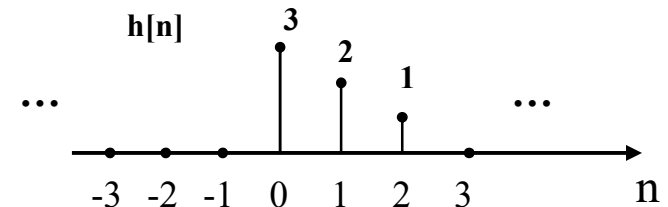
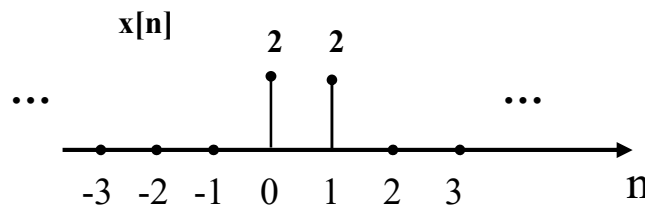


$$y[n]=6\delta[n]+10\delta[n-1]+6\delta[n-2]+ 2\delta[n-3]$$

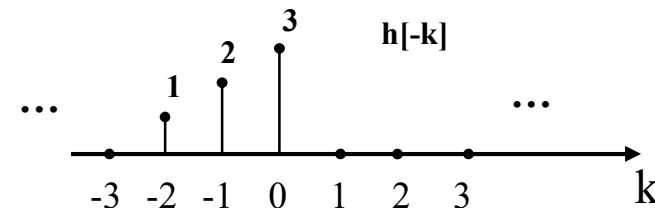
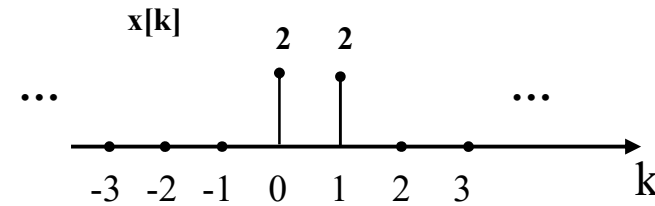


LTI systems: the discrete-time convolution

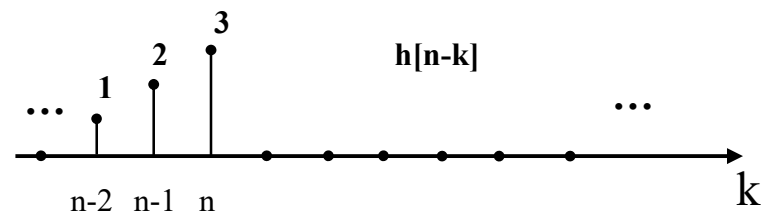
- Method 2: solving by realizing n , we follow a computational procedure similar to the convolution between two continuous-time signals (one of discrete sequences is time-reversed and is shifted from $-\infty$ till $+\infty$, and for each value of the shift, the accumulation of the sample-to-sample product between this sequence and the other –frozen- signal is computed). Using our example:

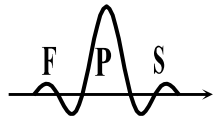


we have:

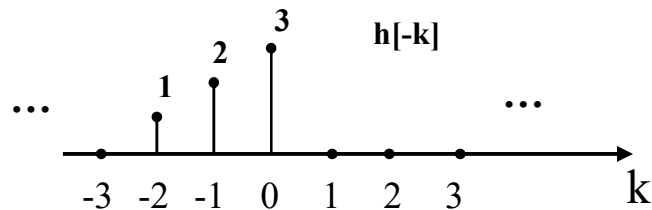
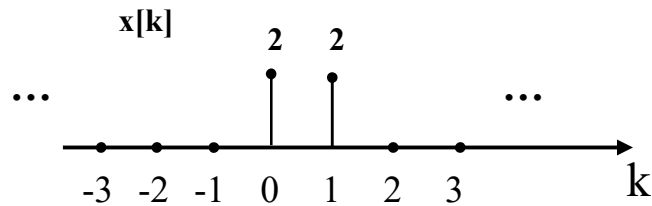


it is clear that
 $n < 0$, $y[n] = 0$. For
 example, for $n = -3$ we have:
 $y[-3] = x[0]h[-3] + x[1]h[-2] = 0$

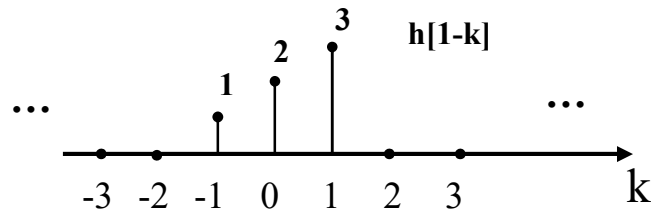




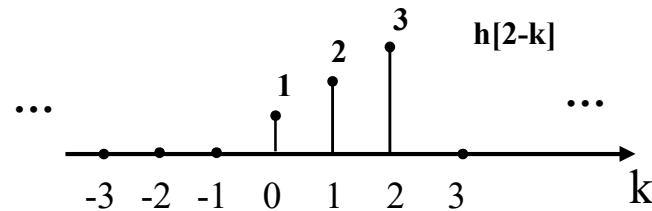
LTI systems: the discrete convolution



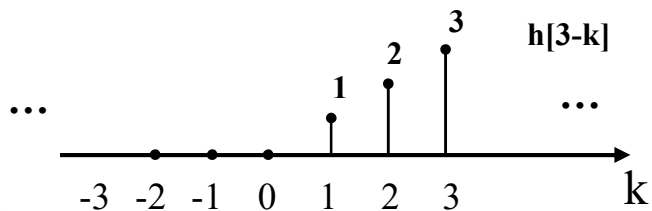
$$y[0] = \sum x[k]h[-k] = x[0]h[0] = 6$$



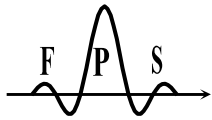
$$y[1] = \sum x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 10$$



$$y[2] = \sum x[k]h[2-k] = x[0]h[2] + x[1]h[1] = 6$$



$$y[3] = \sum x[k]h[3-k] = x[1]h[2] = 2$$

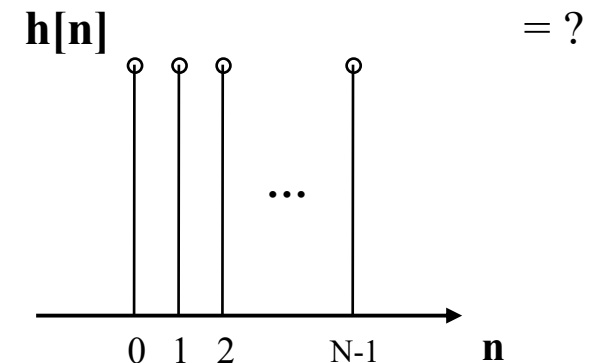
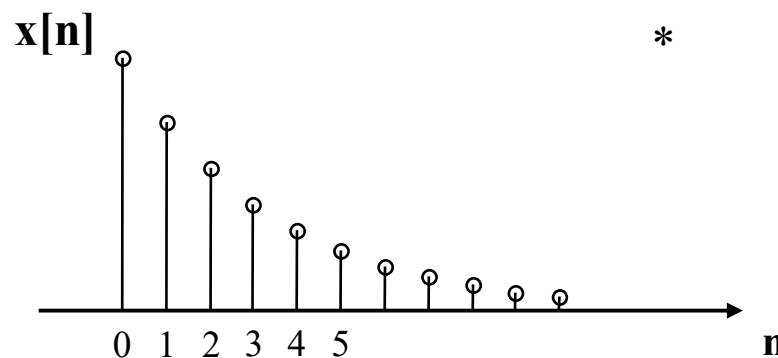


LTI systems: the discrete convolution

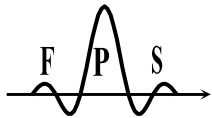
it should be noted that $y[n]=0$ for $n>3$ since $x[k]$ and $h[n-k]$ are not simultaneously different from zero for any value of k .

In summary: $y[n]=6\delta[n]+10\delta[n-1]+6\delta[n-2]+ 2\delta[n-3]$

Another example: $h[n]=u[n]-u[n-N]$ and $x[n]=\alpha^n u[n]$, with $|\alpha|<1$

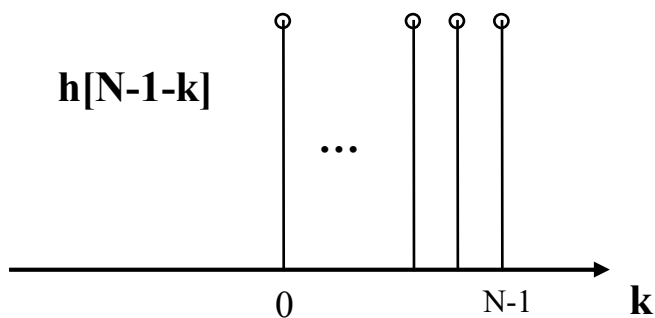
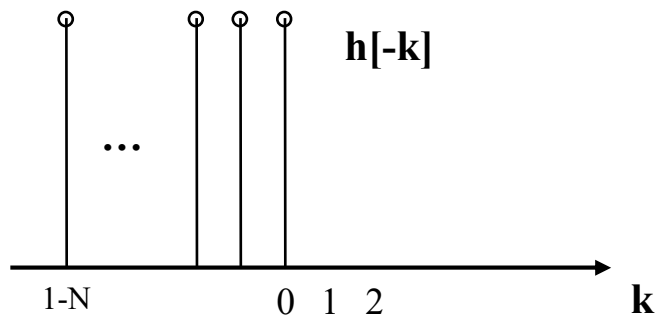
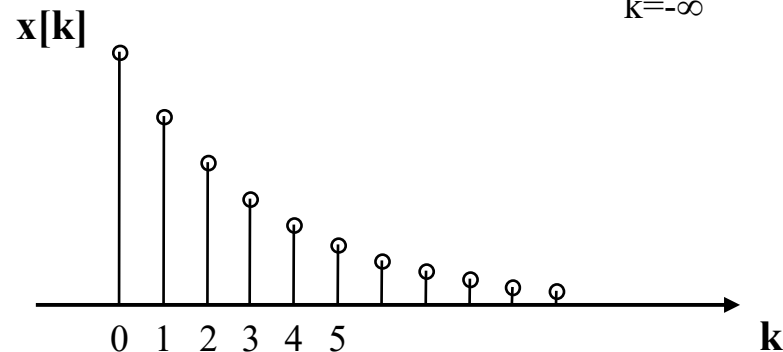


QUESTION: which of the previous two methods is more convenient ?



LTI systems: the discrete convolution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

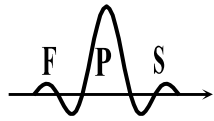


Graphically, it is apparent that we have three intervals for which the $y[n]$ result may be given by a single expression that is valid for all the values of n inside each interval:

Interval 1: $n < 0 \rightarrow y[n] = 0$

Interval 2: $0 \leq n \leq N-1$

Interval 3: $N-1 \leq n$



LTI systems: the discrete convolution

- Interval 1: $n < 0$, $y[n]=0$
- Interval 2: $0 \leq n \leq N-1$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=0}^n \alpha^k u[k] (u[n-k] - u[n-k-N]) = \sum_{k=0}^n \alpha^k$$

$u[n-k] \equiv 1 \text{ for } n-k \geq 0 \Leftrightarrow k \leq n$
 $u[n-k-N] \equiv 1 \text{ for } n-k-N \geq 0 \Leftrightarrow k \leq n-N$

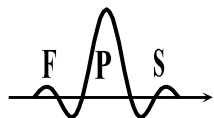
final: $k \geq 0 \ \&\& \ n-N+1 \leq k \leq n \equiv 0 \leq k \leq n$
 (it is easier to conclude graphically)

- Interval 3: $n \geq N-1$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=n-N+1}^n \alpha^k u[k] (u[n-k] - u[n-k-N]) = \sum_{k=n-N+1}^n \alpha^k$$

$u[k] \equiv 1 \text{ for } k \geq 0$
 $\equiv 1 \text{ for } n-N+1 \leq k \leq n$

final: $k \geq 0 \ \&\& \ n-N+1 \leq k \leq n \equiv n-N+1 \leq k \leq n$
 (it is easier to conclude graphically)



LTI systems: the discrete convolution

- Since the sum of M terms of a geometric series is given by the expression:

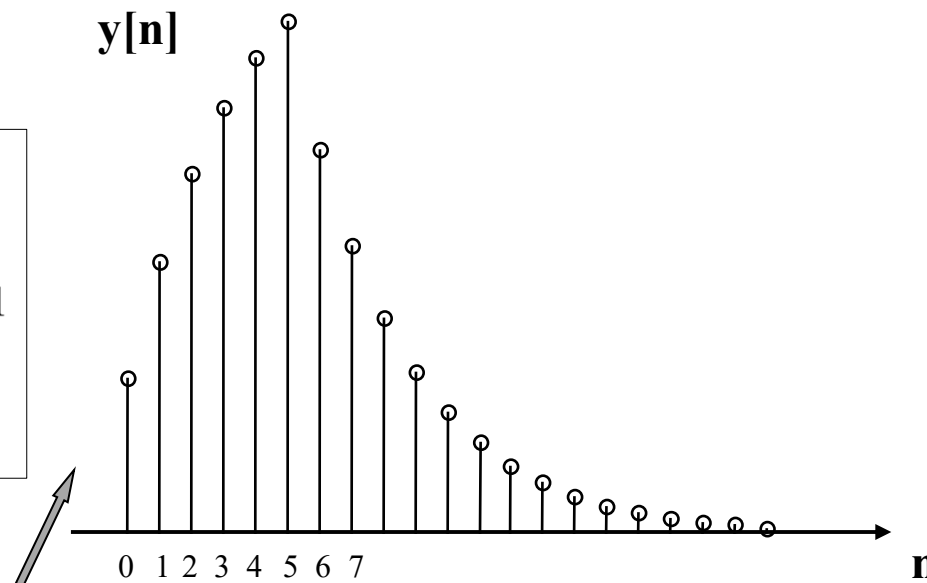
$$G_M = \sum_{k=0}^M \alpha^k = \frac{1 - \alpha^{M+1}}{1 - \alpha}$$

(for an arithmetic series it would be:)

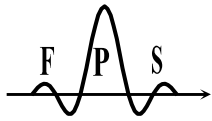
$$A_M = \sum_{k=1}^M k = \frac{M(M+1)}{2}$$

the final result is:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & 0 \leq n \leq N-1 \\ \frac{\alpha^{n-N+1} - \alpha^{n+1}}{1 - \alpha}, & n \geq N-1 \end{cases}$$



In this illustrative case, what is the value of N ?

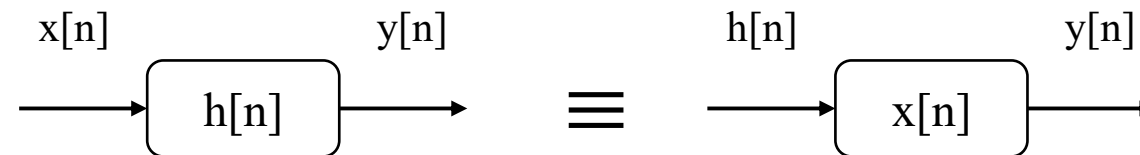


Properties of LTI systems

- Since an LTI system is completely characterized by its impulse response, its properties follow those of the discrete-time convolution

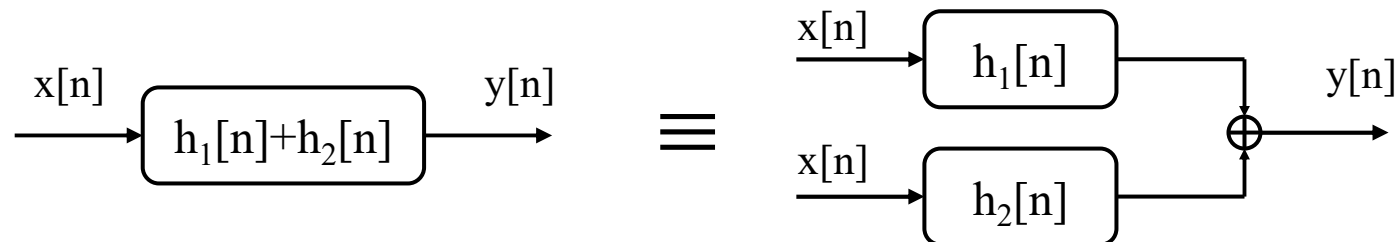
- commutative property:

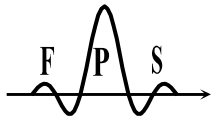
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = h[n] * x[n]$$



- distributive property (of the convolution relative to the sum):

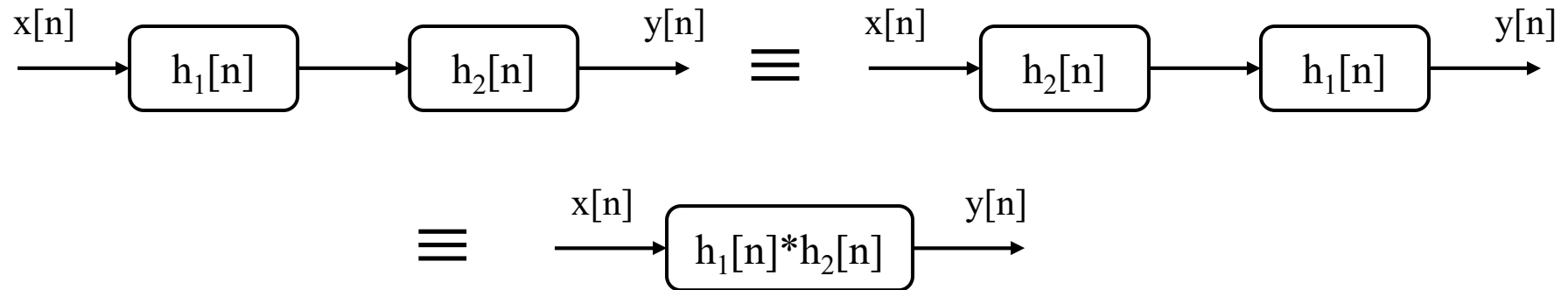
$$y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



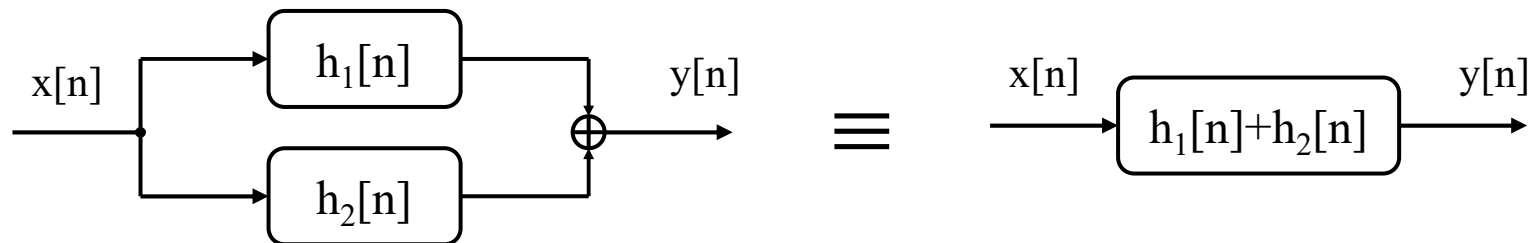


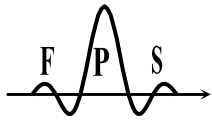
Properties of LTI systems

– series of systems:



– parallel of systems:





Properties of LTI systems

- condition for the stability of an LTI system: if and only if its impulse response is absolutely summable:

$$S = \sum_{k=-\infty}^{+\infty} |h[k]| < \infty \quad (\text{necessary and sufficient condition})$$

- condition for the causality of an LTI system:

$$h[n] = 0, \quad n < 0$$

example: what is the impulse response of each one of the following LTI systems ?

$$y_1[n] = x[n - n_d]$$

$$y_2[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} x[n - k]$$

$$y_3[n] = \sum_{k=-\infty}^n x[k]$$

since $h[n] = y[n]$

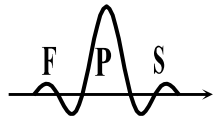
we have:

$$h_1[n] = \delta[n - n_d]$$

$$x[n] = \delta[n]$$

$$h_2[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} \delta[n - k] = \begin{cases} \frac{1}{N_1 + N_2 + 1}, & -N_1 \leq n \leq N_2 \\ 0, & \text{other} \end{cases}$$

$$h_3[n] = \sum_{k=-\infty}^n \delta[k] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0 & n < 0 \end{cases}$$



LTI systems: FIR and IIR systems

- According to the number of non-zero samples of its impulse response, an LTI system may be classified as:

- FIR (finite-duration impulse response): if $h[n]$ has a finite number of non-zero samples

- NOTE 1: FIR systems are always stable
- NOTE 2: a non-causal FIR system may be converted into a causal FIR system by adding a suitable delay

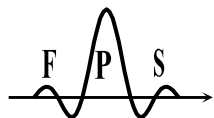
- IIR (infinite-duration impulse response): if $h[n]$ has an infinite number of non-zero samples

- NOTE: IIR systems may be stable, for example:

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1 \quad \rightarrow \quad S = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-\alpha}$$

- NOTE: IIR systems may also be unstable, for example:

$$h[n] = u[n] \quad \rightarrow \quad S = \sum_{n=0}^{\infty} 1 = \infty$$



Constant-coefficient difference equations

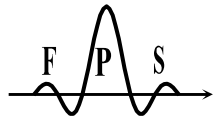
- Consists in an alternative way (relative to the impulse response) to characterize (although not completely) a sub-class of LTI systems (the characterization is only complete if it is added, for example, that the system is causal and starts from rest) by relating a combination of delayed inputs with a combination of delayed outputs, which describes (a realization of) the system:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m] \Leftrightarrow y[n] = -\sum_{k=1}^N \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^M \frac{b_m}{a_0} x[n-m]$$

- NOTE 1: this form emphasizes the recursive nature of the relation: the output is obtained after the input sequence is known and after the previous values of the output sequence are known.
- NOTE 2: if $N=0$, we have:

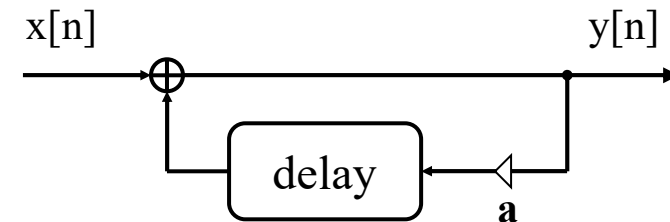
$$y[n] = \sum_{m=0}^M \frac{b_m}{a_0} x[n-m] \Leftrightarrow h[n] = \sum_{m=0}^M \frac{b_m}{a_0} \delta[n-m]$$

which reveals we are dealing with an FIR system, while the more general equation describes an IIR system.



Constant-coefficient difference equations

Example: what is the impulse response of the causal system described by the difference equation: $y[n]=ay[n-1]+x[n]$

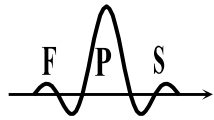


A: considering $x[n]=k\delta[n]$ e admitting that the system starts from rest:

n	x[n]	y[n-1]	y[n]
-1	0	0	0
0	k	0	k
1	0	k	ak
2	0	ak	a ² k
3	0	a ² k	a ³ k
4	0	a ³ k	a ⁴ k
:	:	:	:
n	0	a ⁿ⁻¹ k	a ⁿ k

this means: $h[n]=a^n u[n]$

NOTE: the same system may be described by different difference equations; a specific difference equation is indicative of a specific realization of a discrete system among several possible alternatives (topic to be detailed later on).



Basic signal properties

- Continuous *versus* discrete
 - a continuous-time signal is a real or complex function of one or more independent variables that, most often, are real-valued, e.g. $x_c(t)$
 - the round brackets reinforce that the independent variable is continuous-time
 - a discrete-time signal is a real or complex function of one or more independent variables that can take on integer values only, e.g. $x[n]$
 - the square brackets denote that the enclosed variable is discrete-time

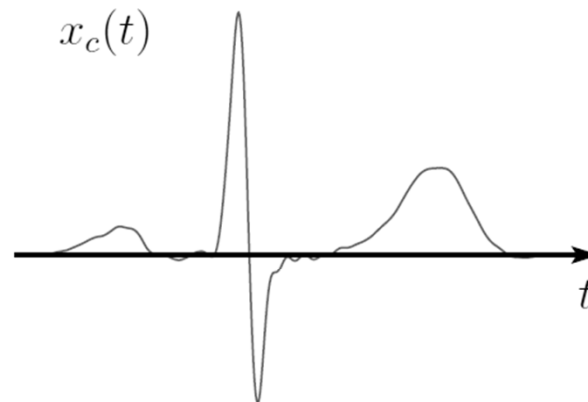


Fig. 1.16 A continuous-time signal representing the bioelectric voltage due to one cycle of the heart beat.

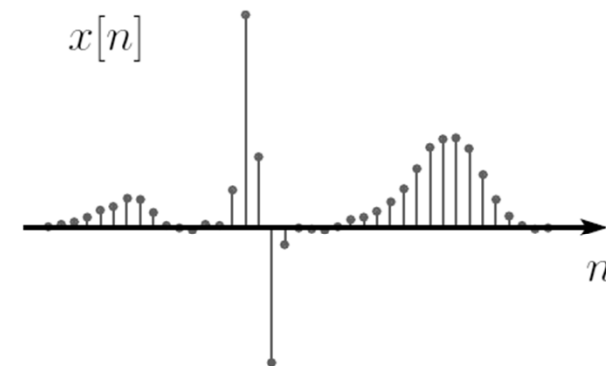
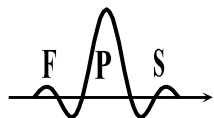


Fig. 1.17 A discrete-time version of the continuous-time signal in Fig. 1.16



Basic signal properties

- Periodic *versus* aperiodic
 - a periodic discrete-time signal is one whose structure or pattern repeats in n for some finite period N , i.e.

$$x[n] = x[n + N], \quad N \in \mathbb{Z} \setminus \{0\}, \quad \forall n \in \mathbb{Z}$$

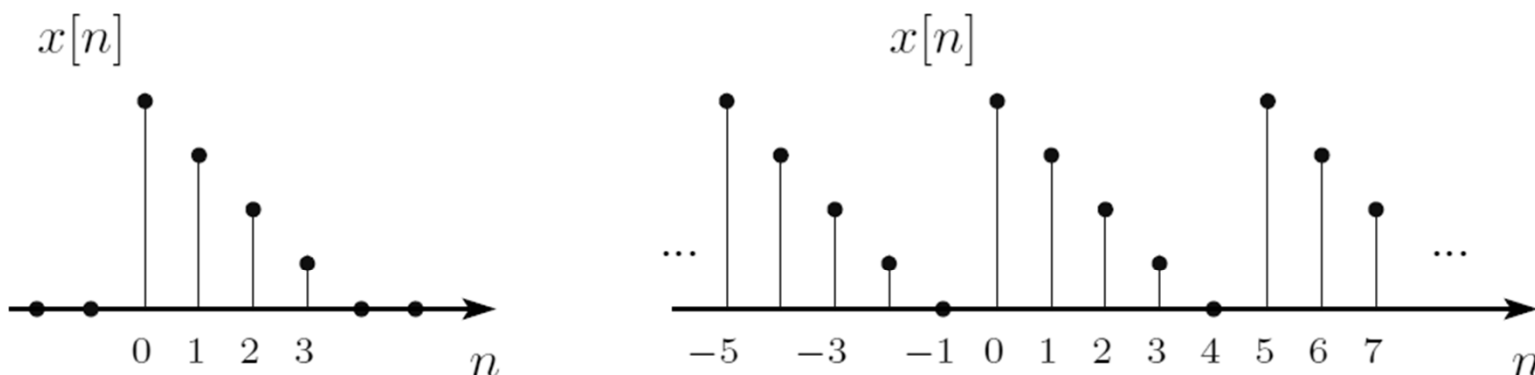
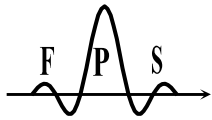


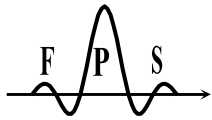
Fig. 1.20 Illustration of an aperiodic discrete sequence (figure on the left) and a periodic discrete sequence whose period is $N = 5$ (figure on the right).

Question: if a periodic signal is obtained by repeating periodically an aperiodic one, what condition makes that the aperiodic signal is recognizable in the periodic signal ?



Basic signal properties

- Deterministic *versus* random
 - a deterministic signal is specified by a mathematical function, an algorithmic procedure, or computational method, which completely determines the signal value at any point in the discrete-time domain n
 - quite often, these deterministic signal representation alternatives are replaced by a lookup table (LUT) when trading computation for memory is desired
 - a notable feature of a deterministic signal is that it is predictable
 - if a signal is deterministic, so are its statistical properties, for example, the probability density function (PDF) of the signal amplitudes
 - a random signal is characterized by unpredictability and uncertainty concerning the value of each sample in a discrete-time sequence
 - instead of being governed by a deterministic rule, the realization of each sample in a random sequence is governed by a probabilistic model underlying the stochastic process that generates that sequence
 - although a random signal is unpredictable at sample level, certain practical assumptions, such as stationarity, make that ‘latent’ probabilistic attributes underlying a random signal, are predictable



Basic signal properties

- Deterministic *versus* random
 - a stationary random signal may exhibit a specific PDF, for example, uniform, of Gaussian
 - these possibilities are illustrated next and have been created using the Matlab functions `rand()` and `randn()`

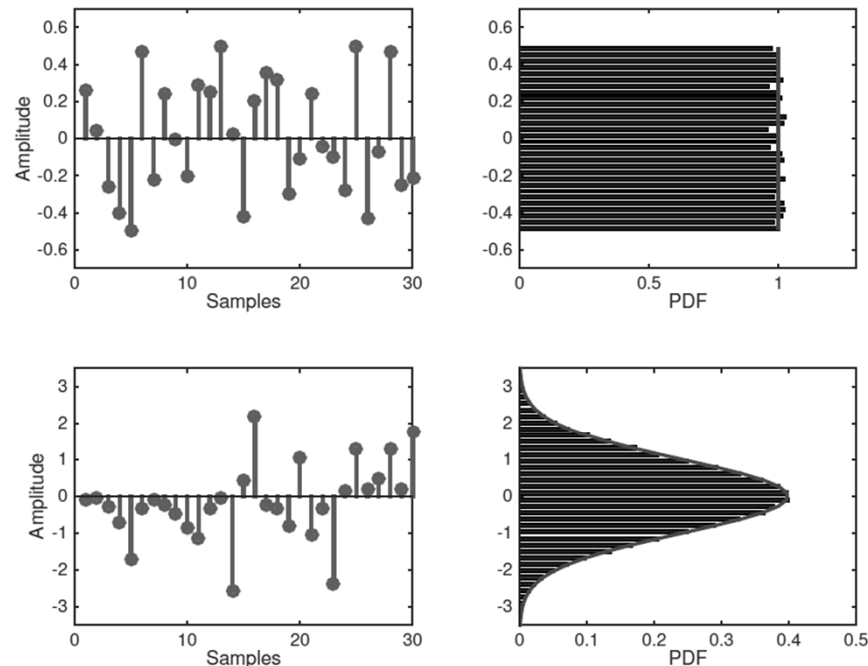
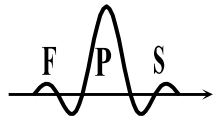


Fig. 1.23 Representation of the first 30 samples of two random sequences, one having a uniform PDF between -0.5 and 0.5 (top left) and another having a Gaussian PDF with unit variance (bottom left). The corresponding PDF are inferred on the right-hand side by means of histograms. The exact PDF models are also represented as a solid (magenta) lines.



Basic signal properties

- Energy *versus* power

- a discrete-time signal $x[n]$ is classified as an energy signal if its energy (E) is finite, i.e. if

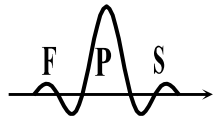
$$E = \sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- any finite-length sequence is always classified as an energy signal as long as its samples have a finite magnitude
- a discrete-time signal $x[n]$ is classified as a power signal if its energy is infinite but a finite result is obtained when the energy of an arbitrary large number of samples is divided by the number of samples

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 < \infty$$

- in the case of an N -periodic signal

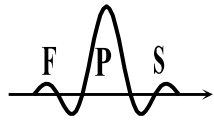
$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 < \infty$$



Basic signal properties

- Sinusoidal sequences with a prescribed SNR
 - frequently, discrete-time test signals need to be generated that have a specific level of noise contamination; most often, these test signals consist of a single complex exponential, or a single sinusoidal sequence, and the noise consists of complex or real-valued Gaussian noise
 - the severity of the noise contamination is objectively determined by the ratio between the average power of the signal, which we represent as P_S , and the average power of the noise, which we represent as P_N
 - that ratio is typically evaluated on a logarithmic scale in tenths of a unit called Bel, which is usually abbreviated to deciBel, or dB; thus, this scale expresses a proportion which is called the Signal-to-Noise Ratio (SNR) and is defined as

$$SNR = 10 \log_{10} \frac{P_S}{P_N}$$



Basic signal properties

- Sinusoidal sequences with a prescribed SNR
 - the following example illustrates the case of a real sinusoid that is contaminated by Gaussian noise according to a prescribed SNR; the result is confirmed numerically in Matlab

Example 1.12. Create in Matlab a real-valued sinusoidal sequence having frequency $\omega_0 = 0.0123$ rad. and magnitude $A = 1$. Create a noise vector such that when it is added to the sinusoidal sequence the resulting SNR is 30 dB. Validate numerically the Matlab code using 10^4 samples.

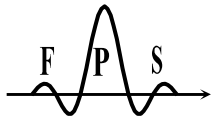
Examples 1.8 and 1.9 have shown that the average power of a real-valued sinusoid (or co-sinusoid) having magnitude A is given by $A^2/2$. Example 1.11 has also shown that the average power of (zero-mean) Gaussian noise is given by σ^2 . Generating a noise vector complying with a desired SNR just involves using the `randn(·)` Matlab function with the appropriate σ parameter. Thus, we find first σ based on (1.55):

$$\sigma = \frac{A}{10^{SNR/20} \sqrt{2}}.$$

After both discrete sequences are created, a numerical validation is easily achieved by estimating the corresponding average power according to Eq. (1.16.4) or Eq. (1.52). A possible Matlab implementation is listed next.

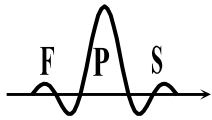
```
N=1E4; n=[0:N-1].';
omega=0.0123; A=1; signal=A*sin(omega*n);
SNR=30; sigma=A/(sqrt(2)*10^(SNR/20));
noise=sigma*randn(N,1);
Ps=mean(abs(signal).^2);
Pn=mean(abs(noise).^2);
10*log10(Ps/Pn)

ans = 30.06
```



The auto-correlation and the cross-correlation

- concept and meaning
 - the auto-correlation and the cross-correlation are two important discrete-time signal processing functions that consist in specific forms of the discrete-time convolution
 - they are designed to evaluate the similarity between two discrete-time signals, or waveforms
 - if the two waveforms are based on the same discrete-time signal, then the function is called auto-correlation whereas if the two waveforms are based on different discrete-time signals, then the function is called cross-correlation

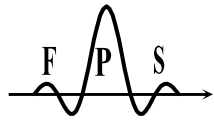


The auto-correlation and the cross-correlation

- definition of the auto-correlation
 - the auto-correlation ($r_x[\ell]$) assesses the similarity between between a finite-energy reference signal, $x[k]$, and the conjugate of its shifted version $x^*[k - \ell]$, where ℓ represents a discrete-time shift, also commonly referred to as *lag*
 - it is obtained as the discrete-time convolution between $x[\ell]$ and the conjugate of its time-reversed version, $x^*[-\ell]$:

$$r_x[\ell] \triangleq x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]x^*[-(\ell - k)] = \sum_{k=-\infty}^{+\infty} x[k]x^*[k - \ell]$$

- The auto-correlation may be regarded as a self-similarity measure for a given discrete-time shift
- if the self-similarity is strong, then the auto-correlation exhibits a high absolute value; conversely, if the self-similarity is weak, then the auto-correlation exhibits a small absolute value, nearing zero in the case of strong dissimilarity

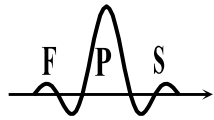


The auto-correlation and the cross-correlation

- definition of the cross-correlation
 - the cross-correlation ($r_{xy}[\ell]$) assesses the similarity between between a finite-energy reference signal, $x[k]$, and $y^*[k - \ell]$ that represents the conjugate of another discrete-time signal, $y[k]$, that is also affected by a lag ℓ
 - it is obtained as the discrete convolution between $x[\ell]$ and $y^*[-\ell]$:

$$r_{xy}[\ell] \triangleq x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]y^*[-(\ell - k)] = \sum_{k=-\infty}^{+\infty} x[k]y^*[k - \ell]$$

- if the similarity between the two waveforms is strong for a given lag, then the cross-correlation function exhibits a high absolute value for that lag, however, if the similarity is weak, then the cross-correlation exhibits a small absolute value; when it nears zero, one may say the signals are approximately uncorrelated



The auto-correlation and the cross-correlation

- auto-correlation and cross-correlation examples
 - a special waveform $x[n]$ is designed such that its auto-correlation consists of a single impulse, i.e. $r_x[\ell] = \delta[\ell]$
 - a second waveform $y[n]$ is generated that consists of a noisy version of $x[n + 3]$, i.e. $y[n] = x[n + 3] + v[n]$, where $v[n]$ is a random sequence that is not correlated to $x[n]$
 - both $r_x[\ell]$ and $r_{xy}[\ell]$ are represented in the following figure

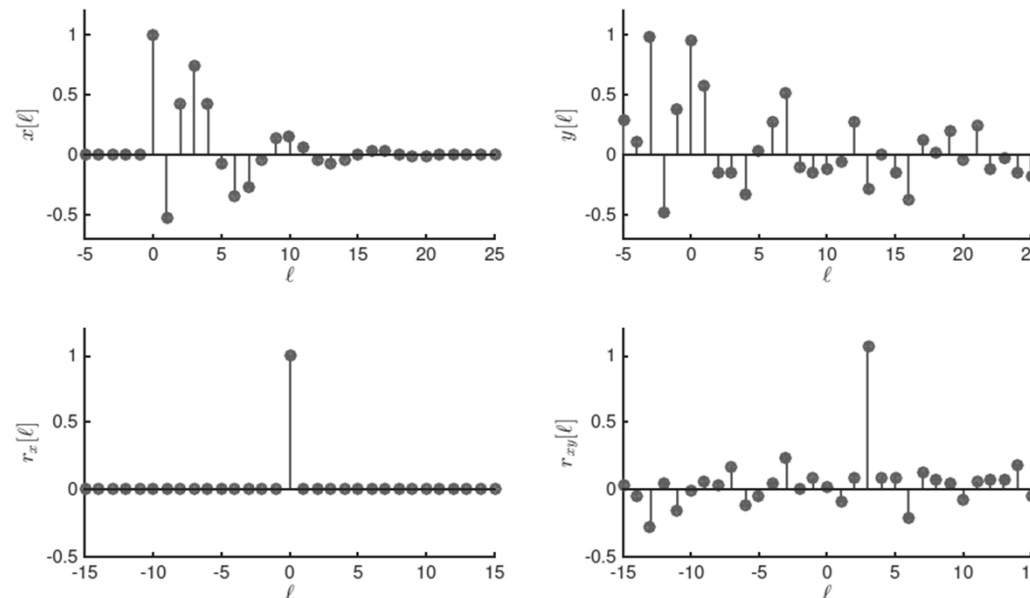
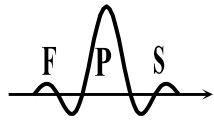


Fig. 1.76 Example of a waveform $x[\ell]$ (top left figure), its auto-correlation function $r_x[\ell]$ (bottom left figure), a waveform $y[\ell]$ consisting of noisy version of $x[\ell + 3]$ (top right figure), and the cross-correlation function $r_{xy}[\ell]$ (bottom right figure).



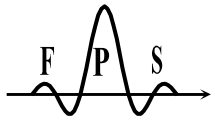
The auto-correlation and the cross-correlation

- auto-correlation and cross-correlation properties
 - in the case of the auto-correlation, it can be easily shown that if the lag is zero, then we obtain the energy of the signal

$$r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = E_x$$

- the auto-correlation is conjugate-symmetric, i.e. $r_x[\ell] = r_x^*[-\ell]$
- the cross-correlation verifies $r_{xy}[\ell] = r_{yx}^*[-\ell]$
 - which does **not** mean conjugate-symmetry
- it can also be shown that $|r_{xy}[\ell]| \leq \sqrt{E_x E_y} = \sqrt{r_x[0] r_y[0]}$
- and, as a particular case, the auto-correlation is upper bounded by the signal energy:

$$|r_x[\ell]| \leq r_x[0] = E_x$$



The auto-correlation and the cross-correlation

- auto-correlation and cross-correlation properties
 - the results in the previous slide can be used to normalize both auto-correlation and cross-correlation functions, which leads to $\rho_x[\ell]$ and $\rho_{xy}[\ell]$:

$$\rho_x[\ell] = \frac{r_x[\ell]}{r_x[0]} , \quad -1 < \rho_x[\ell] \leq 1, \quad \forall \ell \in \mathbb{Z}$$

$$\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_x[0]r_y[0]}} , \quad -1 \leq \rho_{xy}[\ell] \leq 1, \quad \forall \ell \in \mathbb{Z}$$

Question: the previous results have been developed for energy signals, how should they be adapted to power signals ?