Summary

- *The DTFT of the auto-correlation and of the cross-correlation*
  (extended review, week 02)
  - *the DTFT of the auto-correlation*
  - *the DTFT of the cross-correlation*
  - *auto/cross-correlation basic properties*
  - *auto/cross-correlation between output and input of an LSI system*
  - *examples*

- *The Z-Transform of the auto/cross-correlation*
  (extended review, weeks 04/05)
  - *the Z-Transform of the auto-correlation*
  - *the Z-Transform of the cross-correlation*

- *Computing the auto/cross-correlation of finite-length sequences using the DFT*
  - *our starting point*
  - *the cross-correlation using the DFT*
  - *the auto-correlation using the DFT*

- the DTFT of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, x^*[k-\ell]$$

it characterizes the similarity between a sequence and a copy of itself when it is shifted by a lag (ℓ)

considering the DTFT properties

$$x[\ell] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

$$x^*[\ell] \overset{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{-j\omega}\right)$$

$$x[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{-j\omega}\right)$$

$$x^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{j\omega}\right)$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} R_x\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) \cdot X^*\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

where $R_x\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$ is called the spectral density of energy

- the DTFT of the auto-correlation (cont.)
  - the Wiener-Khintchine Theorem: the auto-correlation and the spectral density of energy form a Fourier pair

$$r_x[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R_x\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

thus,

$$r_x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R\left(e^{j\omega}\right) e^{j\omega\ell} d\omega$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum\nolimits_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} R\left(e^{j\omega}\right) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

which reflects the Parseval Theorem

- the DTFT of the cross-correlation

the cross-correlation is defined as (we admit energy signals)

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, y^*[k-\ell]$$

it characterizes the similarity between a sequence and a copy of another sequence when it is shifted by a lag (ℓ)

considering the DTFT properties

$$x[\ell] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

$$y[\ell] \overset{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\omega}\right)$$

$$y^*[\ell] \overset{\mathcal{F}}{\longleftrightarrow} Y^*\left(e^{-j\omega}\right)$$

$$y[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} Y\left(e^{-j\omega}\right)$$

$$y^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} Y^*\left(e^{j\omega}\right)$$

then

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} R_{xy}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) \cdot Y^*\left(e^{j\omega}\right)$$

- auto/cross-correlation basic properties

  - complex-conjugate symmetry (auto-correlation only!)

    $r_x[\ell] = r_x^*[-\ell]$ $\qquad$ $r_{xy}[\ell] = r_{yx}^*[-\ell]$

    (implies that $R_x(e^{j\omega})$ is real-valued)

  - upper bound

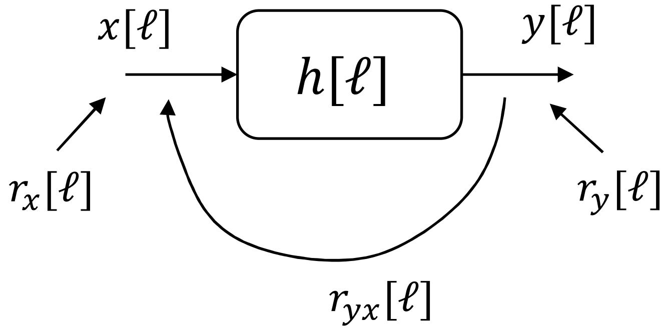
    $|r_x[\ell]| \leq r_x[0]$ $\qquad$ $\left|r_{xy}[\ell]\right| \leq \sqrt{r_x[0] \cdot r_y[0]}$

  - normalized auto-correlation and cross-correlation

    $\rho_x[\ell] = \frac{r_x[\ell]}{r_x[0]}, \quad |\rho_x[\ell]| \leq 1$ $\qquad$ $\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_x[0] \cdot r_y[0]}}, \quad \left|\rho_{xy}[\ell]\right| \leq 1$

The DTFT of the auto-correlation and of the cross-correlation

- auto/cross-correlation between output and input of an LSI system
  - the (quite important !) last two equations are stated without proof



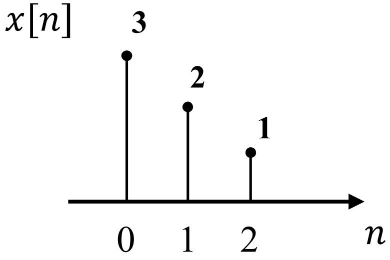
$$y[\ell] = x[\ell] * h[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

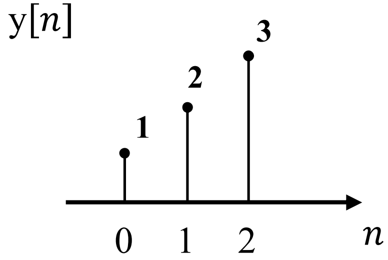
$$r_{yx}[\ell] = r_x[\ell] * h[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R_{yx}(e^{j\omega}) = R_x(e^{j\omega})H(e^{j\omega})$$

$$r_y[\ell] = r_x[\ell] * h[\ell] * h^*[-\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R_y(e^{j\omega}) = R_x(e^{j\omega})\left|H(e^{j\omega})\right|^2$$

- examples

let us admit two finite-length discrete-time signals, $x[n]$ and $y[n]$





it can be easily concluded that

$$x[\ell] = 3\delta[\ell] + 2\delta[\ell-1] + \delta[\ell-2] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X\left(e^{j\omega}\right) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$y[\ell] = \delta[\ell] + 2\delta[\ell-1] + 3\delta[\ell-2] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y\left(e^{j\omega}\right) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$R_x\left(e^{j\omega}\right) = 3e^{j2\omega} + 8e^{j\omega} + 14 + 8e^{-j\omega} + 3e^{-j2\omega} = R_y\left(e^{j\omega}\right),$$ (why ?)

$$R_{xy}\left(e^{j\omega}\right) = 9e^{j2\omega} + 12e^{j\omega} + 10 + 4e^{-j\omega} + e^{-j2\omega}$$

- examples

random sequences may exhibit $r_x[\ell] = \delta[\ell]$

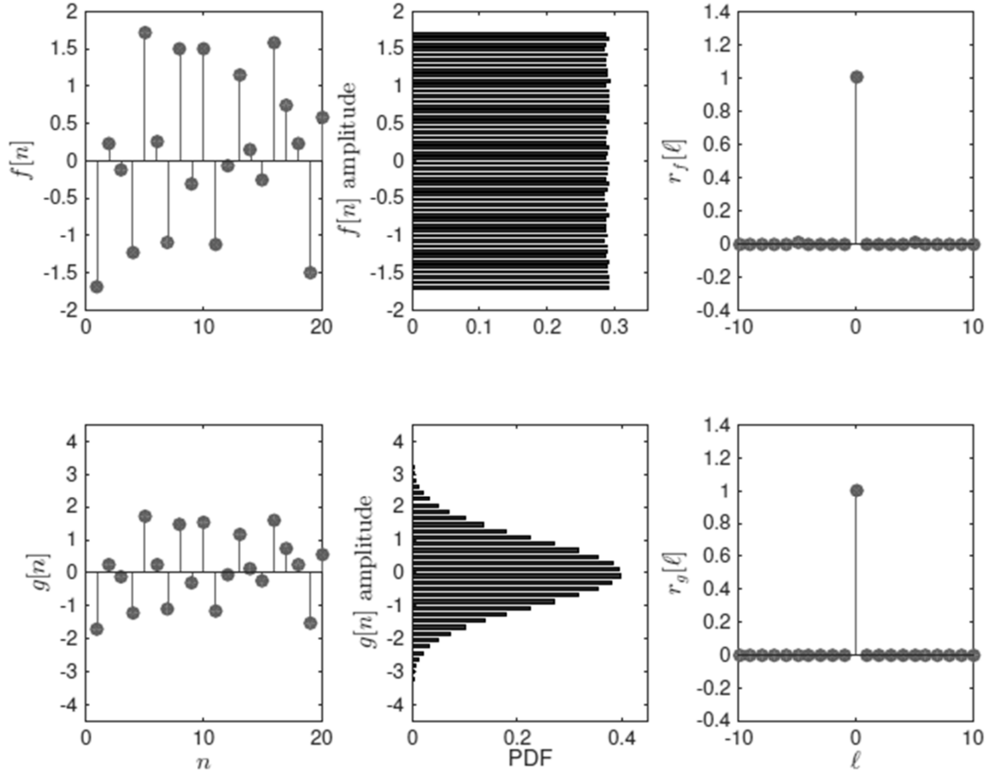


Fig. 1.84 From left to right, the top row of plots represents a few samples of a random sequence ($f[n]$), the flat PDF of its samples amplitudes, and its auto-correlation function $r_f[\ell]$. The bottom row represents the same sequence of plots in the case the random sequence ($g[n]$) has a Gaussian PDF. In both cases, the auto-correlation functions, $r_f[\ell]$ and $r_g[\ell]$, are identical and correspond to the unit impulse.

- examples

deterministic sequences may exhibit $r_x[\ell] = \delta[\ell]$

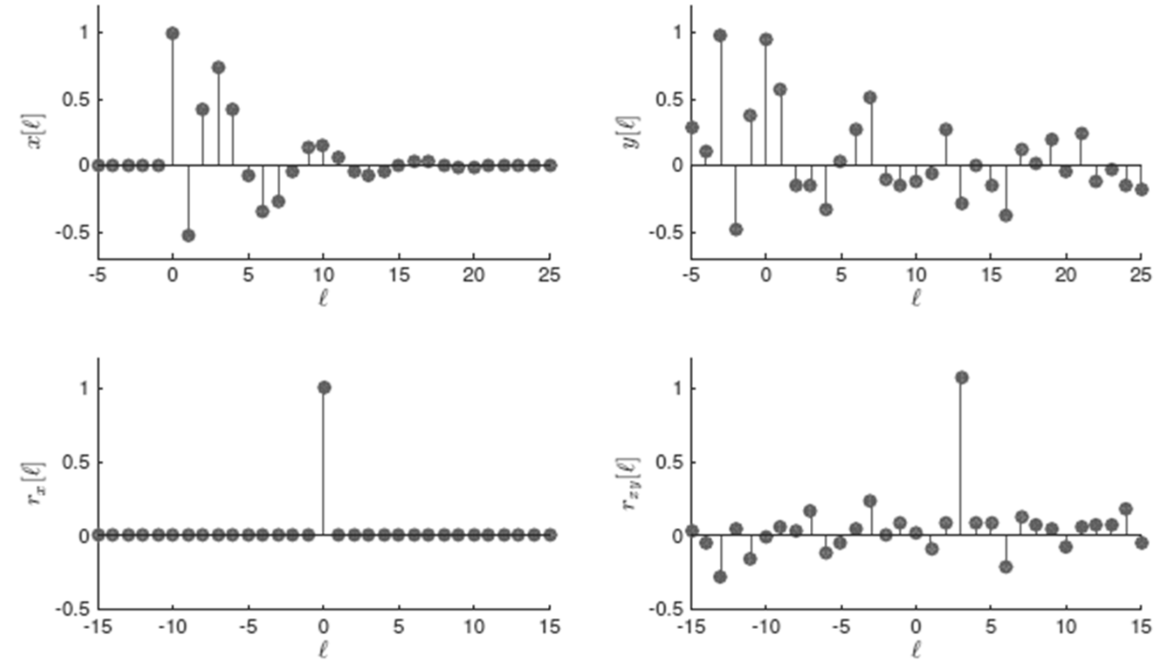


Fig. 1.76 Example of a waveform $x[\ell]$ (top left figure), its auto-correlation function $r_x[\ell]$ (bottom left figure), a waveform $y[\ell]$ consisting of noisy version of $x[\ell+3]$ (top right figure), and the cross-correlation function $r_{xy}[\ell]$ (bottom right figure).

- examples

the auto-correlation is useful to find the period of periodic signals: it is signalled by the *first* local maximum in the $r_x[\ell]$ or $\rho_x[\ell]$ functions ($\ell \neq 0$, why ?)

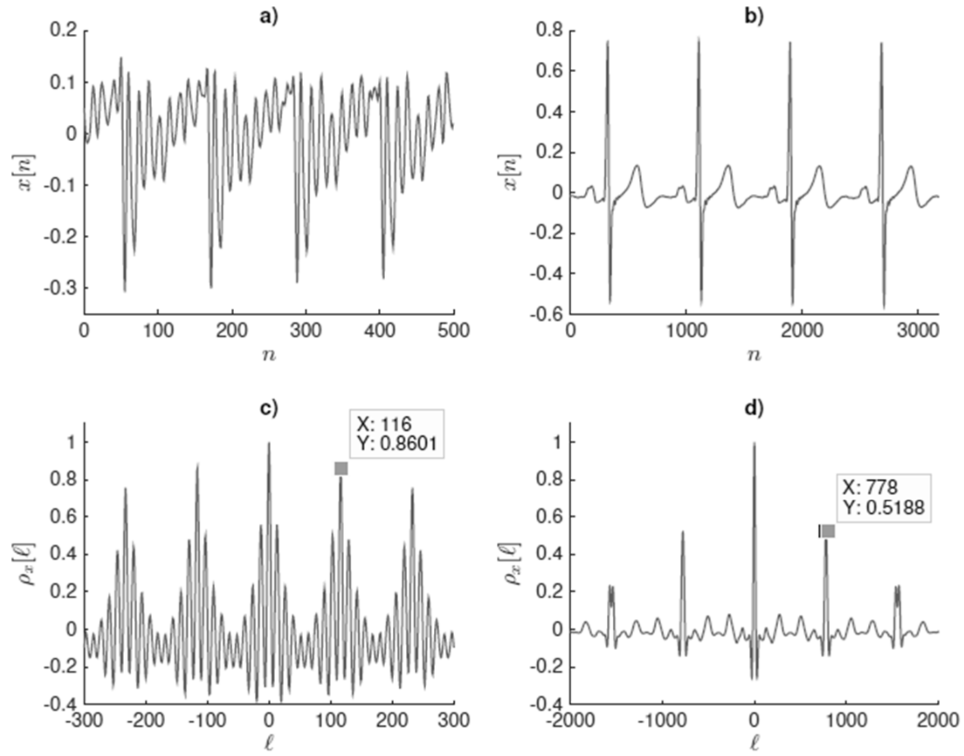


Fig. 1.80 Panel a) represents a segment of a periodic sequence ($x[n]$) corresponding to a vowel signal (the sampling frequency is 22050 Hz), and panel c) represents its normalized auto-correlation function ($\rho_x[\ell]$). Panel b) represents a segment of an ECG signal that has been captured using 1 kHz as the sampling frequency. Panel c) represents it normalized auto-correlation function (see text for details on the auto-correlation analysis).

- the Z-Transform of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]x^*[k-\ell]$$

considering the Z-Transform properties

$$x[\ell] \quad \overset{Z}{\longleftrightarrow} \quad X(z), \qquad RoC = R_x \equiv r_E < |z| < r_D$$

$$x^*[\ell] \quad \overset{Z}{\longleftrightarrow} \quad X^*(z^*), \qquad RoC = R_x$$

$$x[-\ell] \quad \overset{Z}{\longleftrightarrow} \quad X(z^{-1}), \qquad RoC = 1/R_x \equiv 1/r_D < |z| < 1/r_E$$

$$x^*[-\ell] \quad \overset{Z}{\longleftrightarrow} \quad X^*(1/z^*), \quad RoC = 1/R_x$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \quad \overset{Z}{\longleftrightarrow} \quad R_x(z) = X(z) \cdot X^*(1/z^*), \; RoC = R_x \cap 1/R_x$$

where $R_x(z) = X(z) \cdot X^*(1/z^*)$ is called the energy spectrum

- the Z-Transform of the auto-correlation (cont.)
  - the Wiener-Khintchine Theorem: the auto-correlation and the energy spectrum form a Z-Transform pair

$$r_x[\ell] \quad \overset{Z}{\longleftrightarrow} \quad R_x(z) = X(z) \cdot \; X^*(1/z^*)$$

thus,

$$r_x[\ell] = \frac{1}{2\pi j} \oint_C \; R_x(z)\, Z^{\ell-1} dz$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi j} \oint_C \; X(z) \cdot \; X^*(1/z^*)\, Z^{-1} dz$$

which reflects the Parseval Theorem in the Z-domain

- the Z-Transform of the cross-correlation

the cross-correlation is defined as (we admit energy signals)

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, y^*[k-\ell]$$

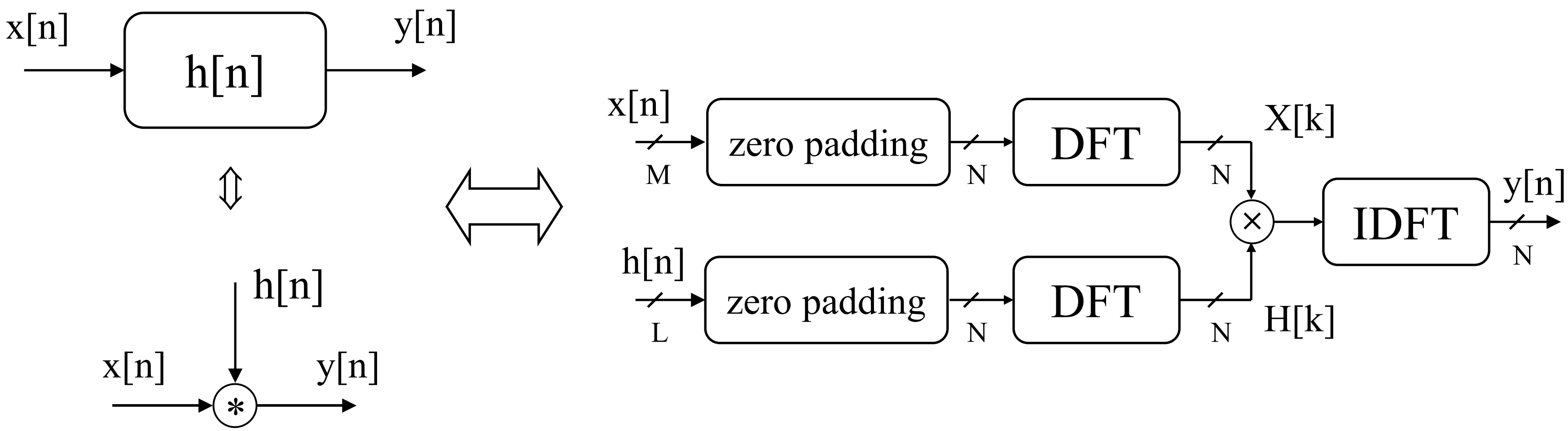
considering the Z-Transform properties

$$x[\ell] \quad \overset{Z}{\longleftrightarrow} \quad X(z), \qquad RoC = R_x$$
$$y[\ell] \quad \overset{Z}{\longleftrightarrow} \quad Y(z), \qquad RoC = R_y$$
$$y^*[\ell] \quad \overset{Z}{\longleftrightarrow} \quad Y^*(z^*), \qquad RoC = R_y$$
$$y[-\ell] \quad \overset{Z}{\longleftrightarrow} \quad Y(z^{-1}), \qquad RoC = 1/R_y$$
$$y^*[-\ell] \quad \overset{Z}{\longleftrightarrow} \quad Y^*(1/z^*), \qquad RoC = 1/R_y$$

then

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \overset{Z}{\longleftrightarrow} R_{xy}(z) = X(z) \cdot Y^*(1/z^*),\ RoC = R_x \cap 1/R_y$$

- our starting point
  - we have seen that if a sequence $x[n]$ has length M, and another sequence $h[n]$ has length L, the linear convolution between them corresponds to a sequence whose length is L+M-1
  - we also have seen that if the signals are zero-padded and made periodic with period N, then the linear result convolution result may also be found using the DFT and its properties as long as **N** ≥ **L+M-1**
  - in this case, the circular convolution yields the same result of the linear convolution according to the following block diagram



- the cross-correlation using the DFT
  - assuming that both sequences, $x[\ell]$ and $y[\ell]$, are suitably zero-padded such that the circular convolution reduces to the linear convolution, then $r_{xy}[\ell]$ can be computed using DFT-based frequency domain processing

$$x[\ell] \overset{DFT}{\longleftrightarrow} X[k]$$

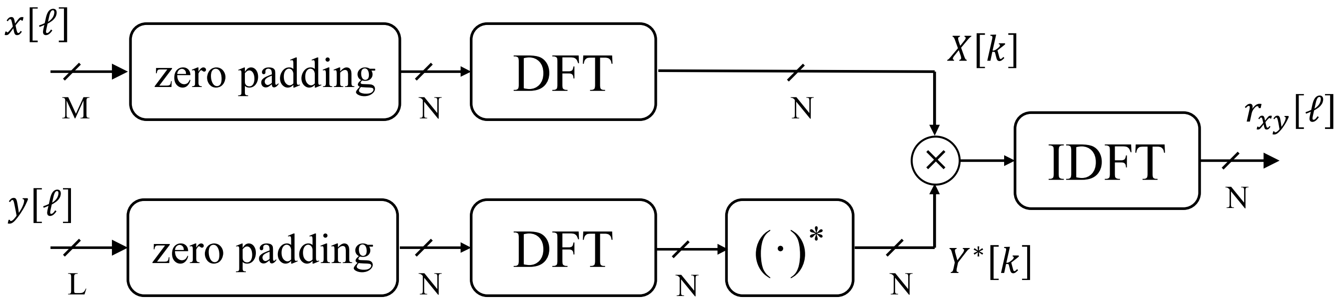
$$y[\ell] \overset{DFT}{\longleftrightarrow} Y[k]$$

$$y^*[\ell] \overset{DFT}{\longleftrightarrow} Y^*[\,(-k)_N]$$

$$y[\,(-\ell)_N] \overset{DFT}{\longleftrightarrow} Y[\,(-k)_N]$$

$$y^*[\,(-\ell)_N] \overset{DFT}{\longleftrightarrow} Y^*[k]$$

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \overset{DFT}{\longleftrightarrow} R_{xy}[k] = X[k] \cdot Y^*[k]$$



- the auto-correlation using the DFT
  - this case is a particular case in the sense that $x[\ell] = y[\ell]$ in the previous slide, which leads to the following (simplified) relationships and block diagram that yields $r_x[\ell]$

$$
\begin{aligned}
x[\ell] &\overset{DFT}{\longleftrightarrow} X[k] \\
x^*[\ell] &\overset{DFT}{\longleftrightarrow} X^*[\,(-k)_N] \\
x[\,(-\ell)_N] &\overset{DFT}{\longleftrightarrow} X[\,(-k)_N] \\
x^*[\,(-\ell)_N] &\overset{DFT}{\longleftrightarrow} X^*[k] \\
r_x[\ell] = x[\ell] * x^*[-\ell] &\overset{DFT}{\longleftrightarrow} R_x[k] = |X[k]|^2
\end{aligned}
$$