FunSP EXAM OF 14 JAN 2026

— SOLUTION TOPICS —

NOTE: these answers are provided with pedagogic concerns.

$\boxed{1.}$ a) A – 1 the zeros that exist on the unit circumference at $z = e^{\pm j\pi/4}$ dictate that the frequency response magnitude is zero-valued for $\omega = \pm\frac{\pi}{4}$ rad.

C – 2 the zeros that exist on the unit circumference at $z = e^{\pm j\frac{\pi}{2}} = \pm j$ dictate that the frequency response magnitude is zero for $\omega = \pm\frac{\pi}{2}$ rad.

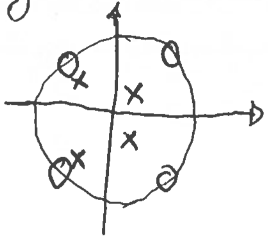
B – 3 similarly to the previous two cases, the zeros at $z = e^{\pm j3\pi/4}$ determine that the frequency response magnitude is zero for $\omega = \pm\frac{3\pi}{4}$ rad.

b) Order: all systems are second-order as they possess two zeros and two poles;

Stability: as all systems are causal, they are also stable because in all cases the unit circumference is located inside the region of convergence;

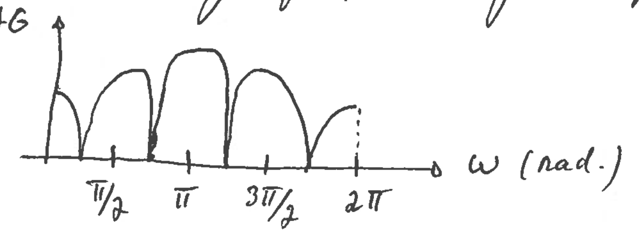
impulse response coefficients: as in all systems the existing poles/zeros occur in complex-conjugate pais (complex-conjugate zeros and complex-conjugate poles) then, it is true for all systems that the coefficients of the impulse response are real-valued.

c) By cascading system A with system B, a 4^{th}-order system results by combining the zeros and poles pertaining to the two systems:

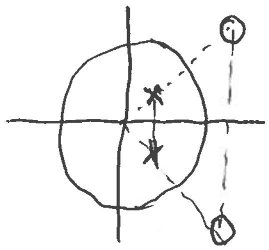


This implies that the frequency response magnitude will be zero-valued for $\omega = \pm\frac{\pi}{4}$ rad and $\omega = \pm\frac{3\pi}{4}$ rad. However, given that in the latter case the poles are closer to the zeros, the notch effect will be narrower than in the former case (i.e., it will be of better quality).

Thus, a sketch of the resulting frequency response magnitude is:



d) By scaling (i.e., by multiplying by a constant factor) the poles and zeros by a constant α, with $|\alpha|>1$, we want to transform the initial zero-pole configuration into a new one in which the zeros are located in the reciprocal-conjugate locations of those of the poles (which corresponds to an all-pass system):



This means that the new zeros are located at $\alpha e^{\pm j\pi/4}$ and the new poles are located at $r\alpha e^{\pm j\pi/4}$.

As we want $\alpha = \frac{1}{r\alpha}$, then $\alpha = \frac{1}{\sqrt{r}}$.

As this reflects the complex-modulation property of the Z-Transform:

$$h_A[n] \xleftrightarrow{Z} H_A(z), \quad RoC_A$$

$$\alpha^n h_A[n] \longleftrightarrow H_A(z/\alpha), \quad |\alpha|\, RoC_A$$

the impulse response modification that reflects the scaling in the Z-domain is:

$$\frac{1}{(\sqrt{r})^n}\, h_A[n], \quad \text{with } r = 0.25.$$

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2.

a) The period of the generated wave is $NT_s = \frac{N}{F_s} = \frac{6}{8000}$ s., which means its frequency is $f = \frac{F_s}{N} = \frac{8000}{6} = 1333.3$ Hz

The normalized frequency is $\omega_0 = \Omega_0 T_s = 2\pi f \frac{1}{F_s} = \frac{2\pi}{N} = \frac{\pi}{3}$ rad.

b) The two pointers should be initialized in such a way that the relative shift, in samples of the sampled wave in the lookup table, amounts to $\frac{\pi}{3}$ rad. As one complete period (2π) is sampled using $N=6$ samples, one single sample represents a $\frac{2\pi}{6} = \frac{\pi}{3}$ phase shift.

Thus, if sine_ptr_L = 0, then setting sine_ptr_R = 1 implements the desired phase shift. Another way of reaching the same conclusion is: we want one wave to be: $\sin n\omega_0$, and another wave to be: $\sin(n\omega_0 + \phi_0)$ with $\phi_0 = \frac{\pi}{3}$, thus, we can write:

$$\sin(n\omega_0 + \phi_0) = \sin((n+n_0)\omega_0) = \sin(n\omega_0 + n_0\omega_0)$$

as $n_0\omega_0 = n_0\frac{\pi}{3} = \frac{\pi}{3} \Rightarrow n_0 = 1$ sample.

c) If sine_ptr_L + 1 is changed to sine_ptr_L + 2 the samples in the lookup table are accessed two times as fast, thus, instead of $\sin(n\omega_0)$, we have $\text{sine}(2n\omega_0) = \text{sine}(n(2\omega_0))$, which means that the frequency (in rad. or Hertz) is multiplied by a factor of two.

d) In this case, N becomes 4, which means that the frequency of the generated wave becomes $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$ rad. or $f = \frac{8000}{4} = 2000$ Hz. It should be noted that although the sample values suggest that a "square" wave is generated, that is not the case as a perfect sinusoidal wave is generated from a succession of 2 positive values and 2 negative values when $N=4$: $\sin \frac{2\pi}{N}(m+\frac{1}{2})\Big|_{N=4;\ m=0,1,2,3}$.

In any case, a "perfect square" wave in the continuous-time domain could never be reconstructed from samples as that would require an infinite bandwith and our D/A reconstruction limits the bandwith to the Nyquist frequency, i.e. to 4 KHz.

$\boxed{3.}$

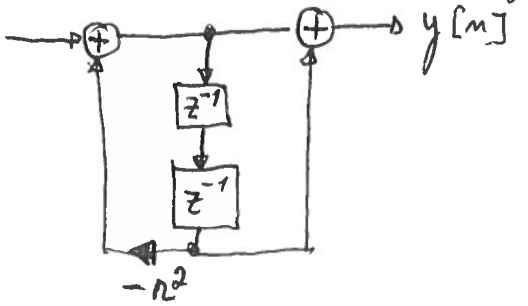
a) In this case we have two zeros $(e^{\pm j\frac{\pi}{2}})$ and two poles $(r e^{\pm j\frac{\pi}{2}})$, therefore, the transfer function of the causal system can be expressed as:

$$H(z) = \frac{\left(1-e^{j\frac{\pi}{2}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{2}}z^{-1}\right)}{\left(1-re^{j\frac{\pi}{2}}z^{-1}\right)\left(1-re^{-j\frac{\pi}{2}}z^{-1}\right)}, \quad |z|>r$$

$$= \frac{1+z^{-2}}{1+r^2z^{-2}}, \quad |z|>r$$

As $H(z) = \frac{Y(z)}{X(z)}$ and, admitting that $1+r^2z^{-2} \neq 0$, we can write $Y(z) = X(z) + z^{-2}X(z) - r^2z^{-2}Y(z)$, which leads to $y[n] = x[n] + x[n-2] - r^2 y[n-2]$.

A possible canonic realization structure is:



b) It is more practical if we start by considering

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) \quad , \quad \text{As } H(e^{j\omega}) = \frac{1+e^{-j2\omega}}{1+r^2e^{-j2\omega}}$$

we obtain:

$$H(e^{j\omega})H^*(e^{j\omega}) = \frac{1+e^{-j2\omega}}{1+r^2e^{-j2\omega}}\,\frac{1+e^{j2\omega}}{1+r^2e^{j2\omega}} = \frac{1+2\cos 2\omega+1}{1+2r^2\cos 2\omega+r^4}$$

$$= \frac{2(1+\cos 2\omega)}{1+2r^2\cos 2\omega+r^4} = \frac{4\cos^2\omega}{1+2r^2\cos 2\omega+r^4} \, ,$$

which leads to

$$|H(e^{j\omega})| = \frac{2|\cos\omega|}{\sqrt{1+2r^2\cos 2\omega+r^4}} \quad ; \text{ if } \omega=0 \text{ or } \omega=\pi \text{ rad.}$$

we obtain: $$|H(e^{j\omega})|\Big|_{\omega=0,\pi} = \frac{2}{\sqrt{1+2r^2+r^4}} = \frac{2}{\sqrt{(1+r^2)^2}} = \frac{2}{1+r^2}$$

c) $F_S = 100\ Hz$

$$x[n] = x_c(t)\Big|_{t=nT_S=\frac{n}{F_S}} = \underbrace{1}_{\omega_0} + \sin 350\pi\frac{n}{100} = 1 + \sin n\underbrace{\frac{7\pi}{2}}_{\omega_1}$$

we have two frequencies:

$\omega_0 = 0$ rad

$$\omega_1 = \frac{7\pi}{2} > \pi \longrightarrow \omega_1 = \frac{7\pi}{2} + K2\pi = \frac{7\pi+K4\pi}{2}\Big|_{K=-2} = -\frac{\pi}{2} \text{ rad}$$

Thus: $x[n] = 1 + \sin(-n\frac{\pi}{2}) = 1 - \sin n\frac{\pi}{2}$

d) As $F(z) = \dfrac{1+z^{-1}}{1+rz^{-1}}$, we have $F(e^{j\omega}) = \dfrac{1+e^{-j\omega}}{1+re^{-j\omega}}$

and: $$F(e^{j\omega})\Big|_{\omega=0} = \frac{1+1}{1+r} = \frac{2}{1+r}$$

$$F(e^{j\omega})\Big|_{\omega=\frac{\pi}{2}} = \frac{1-j}{1-jr} = \frac{\sqrt{2}\,e^{-j\pi/4}}{\sqrt{1+r^2}\,e^{-j\,\mathrm{arctg}\,r}} = \sqrt{\frac{2}{1+r^2}}\,e^{j(\mathrm{arctg}\,r-\frac{\pi}{4})} \, ,$$

which leads to $$y[n] = \frac{2}{1+r} - \sqrt{\frac{2}{1+r^2}}\,\sin\left(n\frac{\pi}{2} + \mathrm{arctg}\,r - \frac{\pi}{4}\right)$$

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As reconstruction is considered ideal, then

$$y[n] = y_c(t)\Big|_{t=\frac{n}{F_S}} = \frac{2}{1+r} - \sqrt{\frac{2}{1+r^2}}\, \sin\left(F_S \frac{\pi}{2}\frac{n}{F_S} + \operatorname{arctg} r - \frac{\pi}{4}\right)$$

$$= \frac{2}{1+r} - \sqrt{\frac{2}{1+r^2}}\, \sin\left(50\pi t + \operatorname{arctg} r - \frac{\pi}{4}\right)\Bigg|_{t=\frac{n}{F_S}}$$

which indicates that:

$$y_c(t) = \frac{2}{1+r} - \sqrt{\frac{2}{1+r^2}}\, \sin\left(50\pi t + \operatorname{arctg} r - \frac{\pi}{4}\right)$$

$\boxed{4.}$

a) The code implements:

$$x[n] \longleftrightarrow X[k]$$

$$y[n] \longleftrightarrow Y[k] = X^*[(-k)_N]$$

which, according to the DFT properties, means that

$$y[n] = x^*[n] \equiv [1 \quad 2 \quad 3 \quad 4-j \quad -j \quad -j]$$

b) The code implements:

$$A[k] = \frac{X[k] + X^*[(-k)_N]}{2} \xleftrightarrow{DFT} \frac{x[n] + x^*[n]}{2} = \mathrm{Re}\{x[n]\} = a[n]$$

$$B[k] = \frac{X[k] - X^*[(-k)_N]}{2} \longleftrightarrow \frac{x[n] - x^*[n]}{2} = j\,\mathrm{Im}\{x[n]\} = b[n]$$

Thus, $\mathrm{ifft}(A) \equiv \mathrm{Re}\{x[n]\} = [1 \quad 2 \quad 3 \quad 4 \quad 0 \quad 0]$

$\mathrm{ifft}(B) \equiv j\,\mathrm{Im}\{x[n]\} = [0 \quad 0 \quad 0 \quad j \quad j \quad j]$

c) The code uses the following DFT property in obtaining $c[n]$:

$$C[k] = A[k] \cdot B[k] \xleftrightarrow{DFT} a[n] \circledast b[n] = c[n]$$

in words: $c[n]$ is the circular convolution between $a[n]$ and $b[n]$, the real part and the imaginary part of $x[n]$, respectively.

Therefore :

$$\begin{array}{ll} & \begin{array}{cccccc} m=0 & m=1 & & & & m=N-1 \end{array} \\ a[m] \equiv & \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

$$b[m] \equiv \begin{bmatrix} 0 & 0 & 0 & j & j & j \end{bmatrix}$$

$$b[(-m)_N] \equiv \begin{bmatrix} 0 & j & j & j & 0 & 0 \end{bmatrix} \longrightarrow \sum_{\ell} = 9j$$

$$b[(1-m)_N] \equiv \begin{bmatrix} 0 & 0 & j & j & j & 0 \end{bmatrix} \longrightarrow \sum_{\ell} = 7j$$

$$b[(2-m)_N] \equiv \begin{bmatrix} 0 & 0 & 0 & j & j & j \end{bmatrix} \longrightarrow \sum_{\ell} = 4j$$

$$b[(3-m)_N] \equiv \begin{bmatrix} j & 0 & 0 & 0 & j & j \end{bmatrix} \longrightarrow \sum_{\ell} = j$$

$$b[(4-m)_N] \equiv \begin{bmatrix} j & j & 0 & 0 & 0 & j \end{bmatrix} \longrightarrow \sum_{\ell} = 3j$$

$$b[(5-m)_N] \equiv \begin{bmatrix} j & j & j & 0 & 0 & 0 \end{bmatrix} \longrightarrow \sum_{\ell} = 6j$$

hence, $c[\ell] = a[\ell] \circledast b[\ell] = \sum_{m=0}^{N-1} a[m]\, b[(\ell-m)_N] \equiv j\begin{bmatrix} 9 & 7 & 4 & 1 & 3 & 6 \end{bmatrix}$

$\boxed{5.}$

a) Given that the sampling frequency is 8 KHz, the Nyquist frequency is 4 KHz.

The relevant difference between the two windows is that the rectangular window causes much more far-end spectral leakage than the Hanning window, which is clearly visible in the spectrograms because the strong concentration of spectral power (dark shading levels) leaks not only to nearby frequency regions, but also to remote frequency regions. Thus :

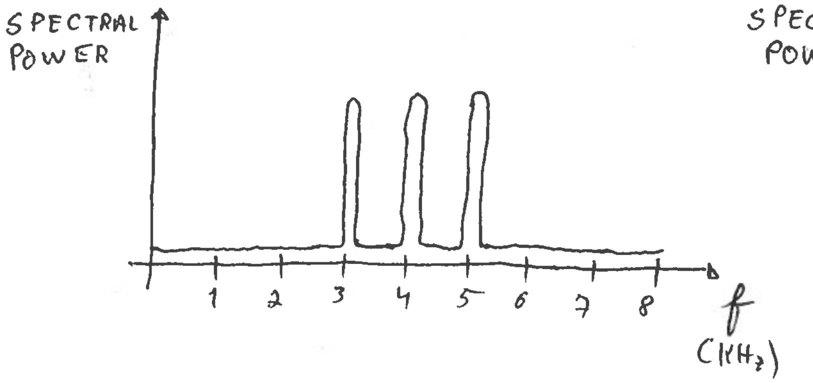
spectrogram A has been generated using the Hanning window, whereas spectrogram B has been generated using the Rectangular window.

b) As spectogram A shows a cleaner representation, it is used to answer this question. The grey-level shading suggests that we have essentially two levels:

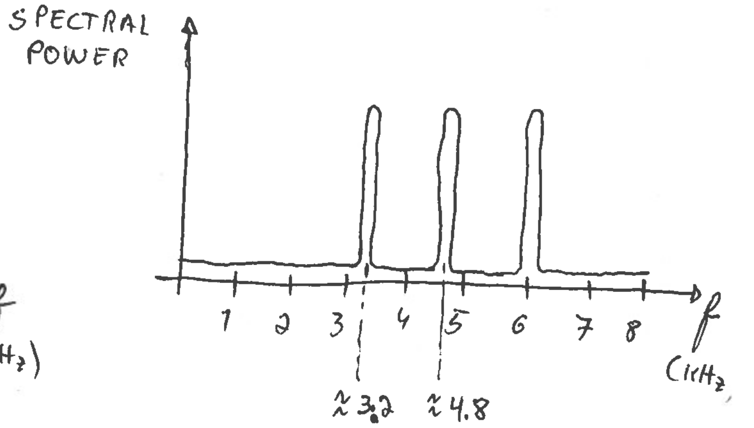
- strong black meaning high spectral density
- light grey meaning low spectral density

Therefore, the requested plausible spectrograms are:

$t = 0.5$ s



$t = 1.2$ s



c) The first aspect to notice in the spectrograms is that the full frequency range, from 0 Hz to the sampling frequency (8 KHz) is shown, instead of just the range between zero and the Nyquist frequency. This is necessary when complex-valued signal components exist because, in this case, the magnitude spectrum is not even-symmetric with respect to 0 Hz and, due to the 2π-periodicity, the magnitude spectrum is also not even-symmetric with respect to the Nyquist frequency. The latter aspect is what facilitates the identification of real-valued signal components:

- a narrow bandwidth signal, probably a sinusoid, whose center frequency is 2 KHz and that is modulated in frequency causing a frequency deviation of about 1 KHz around the mean; the period of the frequency modulation is $0.6 - 0.2 = 0.4$ s. (this is also the time support of the signal)
- a narrow bandwidth signal, probably a sinusoid, whose center frequency starts as 2.5 KHz at

around 0.8 s. and that increases linearly with time up to around 6 KHz at 1.5 s.

Other than these two real-valued signal components, two complex-valued signal components exist that are easily identified because they do not exhibit even-symmetry in the spectrograms with respect to the Nyquist frequency:

- a narrow bandwidth signal, probably a sinusoid, whose center frequency starts as 2 KHz at around 0.1 s. and that increases linearly with time up to around 5 KHz at 0.7 s.

- another narrow bandwidth signal, quite likely a sinusoid, whose center frequency is around 6.5 KHz and that is modulated in frequency (i.e., it is a frequency-modulated sinusoid) causing a deviation of about 1 KHz around the mean; the period of the frequency modulation is 1.2 − 1.0 = 0.2 s. This signal component exists between t = 1.0 s and t = 1.4 s.